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C^*-algebra generated by angular Toeplitz operators on the weighted Bergman spaces over the upper half-plane

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Abstract. We consider the set of all Toeplitz operators acting on the weighted Bergman space over the upper half-plane whose L_∞-symbols depend only on the argument of the polar coordinates. The main result states that the uniform closure of this set coincides with the C^*-algebra generated by the above Toeplitz operators and is isometrically isomorphic to the C^*-algebra of bounded functions that are very slowly oscillating on the real line in the sense that they are uniformly continuous with respect to the arcsinh-metric on the real line.

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1. Introduction

Given a weight parameter $\lambda \in (-1, +\infty)$, we introduce the following standard measure on the upper half-plane Π:

$$d\mu_\lambda(z) = 2^\lambda(\lambda + 1)r^{\lambda+1} \sin^\lambda \theta \, drd\theta, \qquad z = re^{i\theta}.$$

The weighted Bergman space $\mathcal{A}^2_\lambda(\Pi)$ is the subspace of $L_2(\Pi, d\mu_\lambda)$ consisting of all analytic functions. It is well known that $\mathcal{A}^2_\lambda(\Pi)$ is a closed subspace of $L_2(\Pi, d\mu_\lambda)$ (see for example [19, 21]), and that the orthogonal *Bergman projection* B_λ from $L_2(\Pi, d\mu_\lambda)$ onto $\mathcal{A}^2_\lambda(\Pi)$ has the form $(B_\lambda f)(z) = \langle f, K_{z,\lambda}\rangle$, where the function $K_{z,\lambda}\colon \Pi \to \Pi$, the so-called *Bergman kernel* at a point z, is given by the formula

$$K_{z,\lambda}(w) = \left(\frac{i}{w - \overline{z}}\right)^{\lambda+2} \qquad w \in \Pi. \tag{1.1}$$

Given $g \in L_\infty(\Pi)$, the *Toeplitz operator* T_g with defining symbol g and acting on the weighted Bergman space $\mathcal{A}^2_\lambda(\Pi)$ is given by $T_g f = B_\lambda(gf)$. Unfortunately practically nothing can be said on the properties of Toeplitz operators with general L_∞-symbols; though some general results are collected, for example, in [21]. The common strategy here is to study Toeplitz operators with symbols from certain special subclasses of L_∞.

The most complete results are obtained for the families of symbols that generate commutative C^*-algebras of Toeplitz operators. They were described in a series of papers summarized in the book [19], see also [9]. These families of defining symbols lead to the following three model cases: *radial* symbols, functions on the unit disk depending only on $|z|$, *vertical* symbols, functions on the upper half-plane depending on $\operatorname{Im} z$, and *angular* symbols defined on the upper half-plane and depending only on $\arg z$. In each one of these three cases, the Toeplitz operators admit an explicit diagonalization, i.e. there exists an isometric isomorphism that transforms all Toeplitz operators of the selected type to the multiplication operators by some specific functions (we call them *spectral functions*, in the radial case they are just the eigenvalue sequences). Of course, such a diagonalization immediately reveals all the main properties of the corresponding Toeplitz operators.

Then the next natural problem emerges: give an explicit and independent description of the class of spectral functions (and of the algebra generated by them) for each one of the above three cases. First step in this direction was made by Suárez [17, 18]. He proved that the eigenvalue sequences of Toeplitz operators with bounded *radial* symbols form a dense subset in the ℓ_∞-closure of the class d_1 of bounded sequences $(\sigma_k)_{k=1}^\infty$ satisfying

$$\sup_k (k+1)|\sigma_{k+1} - \sigma_k| < +\infty.$$

As a consequence, the C^*-algebra generated by Toeplitz operators with bounded radial symbols is isometrically isomorphic to this ℓ_∞-closure of d_1. The results of Suárez have been complemented and generalized to the weighted Bergman space on the unit ball in [1, 2, 10, 15]. The above ℓ_∞-closure of d_1 was characterized in [10]. As it turned out, this closure coincides with the C^*-algebra $\mathrm{VSO}(\mathbb{N})$ of bounded functions (sequences) $\mathbb{N} \to \mathbb{C}$ that are uniformly continuous with respect to the logarithmic metric $|\ln(j) - \ln(k)|$. Surprisingly this class of sequences was already introduced by Schmidt [16, § 9] in the beginning of the 20th century in connection with Tauberian theory. It is worth mentioning that the above description shows the position of the radial Toeplitz operators amongst all bounded radial operators (the set of which is isomorphic to $\ell_\infty(\mathbb{N})$).

The papers [11, 12] continue this program and give a description of the commutative algebra generated by Toeplitz operators with bounded *vertical* symbols. The result states that their spectral functions form a dense subset in the C^*-algebra $\mathrm{VSO}(\mathbb{R}_+)$ of *very slowly oscillating functions* on $\mathbb{R}_+$, i.e. the bounded functions $\mathbb{R}_+ \to \mathbb{C}$ that are uniformly continuous with respect to the metric $|\ln(x) - \ln(y)|$. This, in particular, means that the C^*-algebra generated by Toeplitz operators with bounded vertical symbols is isometrically isomorphic to $\mathrm{VSO}(\mathbb{R}_+)$.

This paper is devoted to the last remaining model case of *angular* symbols, and completes thus the intrinsic description of the commutative C^*-algebras generated by Toeplitz operators with bounded symbols for each one of the three model classes.

A function $g \in L_\infty(\Pi)$ is said to be *homogeneous of order zero* or *angular* if for every $h > 0$ the equality $g(hz) = g(z)$ holds for a.e. $z \in \Pi$, or, equivalently, if there exists a function a in $L_\infty(0, \pi)$ such that $g(z) = a(\arg z)$ for a.e. z in Π. We denote by $\mathcal{A}_\infty$ this class of functions, and introduce the set $\mathrm{T}_\lambda(\mathcal{A}_\infty)$ of all Toeplitz operators acting on $\mathcal{A}^2_\lambda(\Pi)$ with defining symbols in $\mathcal{A}_\infty$.

As was shown in [8], the uniraty operator $R_\lambda : \mathcal{A}^2_\lambda(\Pi) \to L_2(\mathbb{R})$, where

$$(R_\lambda \varphi)(x) = \frac{1}{\sqrt{2^{\lambda+1}(\lambda+1)\, c_\lambda(x)}} \int_\Pi (\overline{z})^{-ix-(\frac{\lambda+2}{2})}\, \varphi(z)\, d\mu_\lambda(z), \quad x \in \mathbb{R}, \tag{1.2}$$

with

$$c_\lambda(x) = \int_0^\pi e^{-2x\theta} \sin^\lambda \theta\, d\theta, \quad x \in \mathbb{R}. \tag{1.3}$$

diagonalizes each Toeplitz operator T_g with angular symbol $g(z) = a(\arg z)$; that is $R_\lambda T_g R^*_\lambda = \gamma_{a,\lambda} I$, where the spectral function $\gamma_{a,\lambda} : \mathbb{R} \to \mathbb{C}$ is given by

$$\gamma_{a,\lambda}(x) = \frac{1}{c_\lambda(x)} \int_0^\pi a(\theta)\, e^{-2x\theta} \sin^\lambda \theta\, d\theta, \quad x \in \mathbb{R}. \tag{1.4}$$

In particular, this implies that the algebra generated by $\mathrm{T}_\lambda(\mathcal{A}_\infty)$ is isometrically isomorphic to the function algebra generated by

$$\Gamma_\lambda = \{\gamma_{a,\lambda} : \ a \in L_\infty(0, \pi)\}. \tag{1.5}$$

In the paper we describe explicitly this C^*-algebra. We denote by $\mathrm{VSO}(\mathbb{R})$ the C^*-algebra of *very slowly oscillating* functions on the real line [7], which consists of all bounded functions that are uniformly continuous with respect to the *arcsinh-metric*

$$\rho(x, y) = |\operatorname{arcsinh} x - \operatorname{arcsinh} y|, \quad x, y \in \mathbb{R}. \tag{1.6}$$

The main result of the paper (Theorem 4.9) states that the uniform closure of Γ_λ coincides with the C^*-algebra $\mathrm{VSO}(\mathbb{R})$. As a consequence, the C^*-algebra $\mathcal{T}_\lambda(\mathcal{A}_\infty)$ generated by the set $\mathrm{T}_\lambda(\mathcal{A}_\infty)$ coincides just with the closure of the set of its initial generators, and is isometrically isomorphic to $\mathrm{VSO}(\mathbb{R})$. Note that the result does not depend on a value of the weight parameter $\lambda > -1$.

As a by-product of the main result, we show that the closure of the set $\mathrm{T}_\lambda(\mathcal{A}_\infty)$ in the strong operator topology coincides with the C^*-algebra of all angular operators.

With this paper we finish the explicit descriptions of the above mentioned commutative C^*-algebras of Toeplitz operators on the unit disk and upper half-plane. In all three cases the spectral functions oscillate at infinity with the logarithmic speed. The C^*-algebras $\mathrm{VSO}(\mathbb{R}_+)$ and $\mathrm{VSO}(\mathbb{R})$ corresponding to the vertical and angular cases, respectively, are isometrically isomorphic (via the change of variables $v \mapsto \sinh(\ln(v))$), and both of them are isometrically isomorphic to the C^*-algebra of bounded functions on $\mathbb{R}$ uniformly continuous with respect to the usual metric (via the changes of variables $v \mapsto \ln(v)$ and $v \mapsto \operatorname{arcsinh}(v)$, respectively). The sequences from $\mathrm{VSO}(\mathbb{N})$ are nothing but the restrictions to $\mathbb{N}$ of the functions from $\mathrm{VSO}(\mathbb{R}_+)$.

Note that the proof in the vertical case was the simplest one because the corresponding spectral functions $\gamma^v_{a,\lambda}$ admit representations in terms of the Mellin convolutions, and the result about density was obtained just by using a convenient Dirac sequence. Unfortunately, in the angular case this simple approach does not work.

The key idea of the proof presented in this paper is to approximate functions from $\mathrm{VSO}(\mathbb{R})$ by $\gamma^v_{a,\lambda}$ near $+\infty$ and $-\infty$. After that, the problem is reduced to the approximation of $C_0(\mathbb{R})$ functions by appropriate $\gamma_{a,\lambda}$; the latter problem is solved using the duality and the analyticity arguments (Theorem 4.8).

Section 2 contains criteria for angular and angular Toeplitz operators. In Section 3 it is proved that the functions of the class Γ_λ are Lipschitz continuous with respect the metric ρ. Finally, Section 4 is dedicated to the proof of the density.

2. Angular Toeplitz Operators

Let $\mathcal{B}(\mathcal{A}^2_\lambda(\Pi))$ be the algebra of all linear bounded operators acting on the Bergman space $\mathcal{A}^2_\lambda(\Pi)$. Given $h \in \mathbb{R}_+$, let $D_{h,\lambda} \in \mathcal{B}(\mathcal{A}^2_\lambda(\Pi))$ be the *dilation* operator defined by

$$D_{h,\lambda} f(z) = h^{\frac{\lambda+2}{2}} f(hz). \tag{2.1}$$

An operator $V \in \mathcal{B}(\mathcal{A}^2_\lambda(\Pi))$ is said to be *angular* or *invariant under dilations* if it commutes with all dilation operators. We denote by $\mathfrak{A}_\lambda$ the set of all angular operators:

$$\mathfrak{A}_\lambda := \left\{ V \in \mathcal{B}(\mathcal{A}^2_\lambda(\Pi)) \colon \quad \forall h \in \mathbb{R}_+ \quad D_{h,\lambda} V = V D_{h,\lambda} \right\}. \tag{2.2}$$

For each $h > 0$ the operator $D_{h,\lambda}$ can be diagonalized by the unitary operator R_λ given by (1.2).

$$\begin{aligned}
(R_\lambda D_{h,\lambda}\varphi)(x) &= \frac{1}{\sqrt{2^{\lambda+1}(\lambda+1)c_\lambda(x)}} \int_\Pi (\overline{z})^{-ix-\left(\frac{\lambda+2}{2}\right)} h^{\frac{\lambda+2}{2}} \varphi(hz)\, d\mu_\lambda(z) \\
&= \frac{h^{ix}}{\sqrt{2^{\lambda+1}(\lambda+1)c_\lambda(x)}} \int_\Pi (\overline{z})^{-ix-\left(\frac{\lambda+2}{2}\right)} \varphi(z)\, d\mu_\lambda(z) \\
&= h^{ix}(R_\lambda\varphi)(x) = (M_{E_h} R_\lambda \varphi)(x),
\end{aligned}$$

where M_{E_h} is the multiplication operator by the function $E_h(x) = h^{ix}$. This clearly forces

$$R_\lambda D_{h,\lambda} R^*_\lambda = M_{E_h}, \quad h \in \mathbb{R}_+. \tag{2.3}$$

Recall that the *Berezin transform* [3, 4] of an operator $V \in \mathcal{B}(\mathcal{A}^2_\lambda(\Pi))$ is the function $\widetilde{V} \colon \Pi \to \mathbb{C}$ defined by

$$\widetilde{V}(z) := \frac{\langle VK_{z,\lambda}, K_{z,\lambda}\rangle}{\langle K_{z,\lambda}, K_{z,\lambda}\rangle}, \tag{2.4}$$

where $K_{z,\lambda}$ is the Bergman kernel (1.1). The following theorem gives a criterion for an operator to be angular and is analogous to the Zorboska result [20] for radial operators and Herrera Yañez, Maximenko, Vasilevski result [12] for vertical operators. For the unweighted Bergman space, this criterion was established in [7].

Theorem 2.1 (criterion for angular operators).
Let $V \in \mathcal{B}(\mathcal{A}^2_\lambda(\Pi))$. The following conditions are equivalent:
(i) $V \in \mathfrak{A}_\lambda$,
(ii) $R_\lambda V R_\lambda^* M_{E_h} = M_{E_h} R_\lambda V R_\lambda^*$ *for all* $h \in \mathbb{R}_+$,
(iii) *there exists* $\phi \in L_\infty(\mathbb{R})$ *such that* $V = R_\lambda^* M_\phi R_\lambda$,
(iv) *the Berezin transform* $\widetilde{V}$ *depends on* $\arg z$ *only.*

Proof. The proof follows the arguments of [7, Theorem 2.2]. Here we only present few formulas that justify the implication (iii)⇒(iv). For every point $w = \rho\, e^{i\beta}$,

$$(R_\lambda K_{w,\lambda})(x) = \frac{e^{\left(i\left(\frac{\lambda+2}{2}\right)-x\right)\beta}}{\rho^{\left(\frac{\lambda+2}{2}\right)+ix}\,\sqrt{2^{\lambda+1}(\lambda+1)c_\lambda(x)}}, \qquad x \in \mathbb{R}.$$

Therefore, if the operator V is diagonalized by R_λ (item (iii)), then its Berezin transform can be written in terms of the spectral function ϕ and depends only on the polar angle β of w:

$$\widetilde{V}(w) = \frac{1}{K_{w,\lambda}(w)} \int_{\mathbb{R}} \phi(x)\, |R_\lambda K_{w,\lambda}(x)|^2\, dx = \frac{2\sin^{\lambda+2}\beta}{\lambda+1} \int_{\mathbb{R}} \frac{\phi(x)\, e^{-2x\beta}}{c_\lambda(x)}\, dx. \qquad \square$$

The next lemma provides a simple criterion of angular functions.

Lemma 2.2 (criterion for a function to be angular). *Let $g \in L_\infty(\Pi)$. Then the following two conditions are equivalent:*

(a) *for every $h \in \mathbb{R}_+$, the equality $g(hz) = g(z)$ holds for a.e. z in Π,*
(b) *there exists a in $L_\infty(0,\pi)$ such that $g(z) = a(\arg z)$ for a.e. z in Π.*

Proof. This technical result is contained in the proof of Proposition 3.1 in [7]. □

Corollary 2.3. *Given $g \in L_\infty(\Pi)$, the Toeplitz operator T_g is angular if and only if g is angular.*

Proof. Follows easily from Lemma 2.2, Theorem 10.4.16 in [19] and the criterion of angular operators (Theorem 2.1). □

3. Very slowly oscillating functions on the real line

In this section we prove that Γ_λ is a proper subset of $\mathrm{VSO}(\mathbb{R})$. We start with a formal definition of the class $\mathrm{VSO}(\mathbb{R})$.

Definition 3.1. Let $f\colon \mathbb{R} \to \mathbb{C}$. The function $\Omega_{\rho,f} : [0,+\infty] \longrightarrow [0,+\infty]$ defined by

$$\Omega_{\rho,f}(\delta) := \sup\{|f(x) - f(y)|:\ x, y \in \mathbb{R},\ \rho(x,y) \le \delta\}. \tag{3.1}$$

is called the *modulus of continuity* of f with respect to the arcsinh-metric ρ, see (1.6).

Definition 3.2. Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a bounded function. We say that f is *very slowly oscillating* if it is uniformly continuous with respect to ρ, i.e. if $\lim_{\delta\to 0} \Omega_{\rho,f}(\delta) = 0$.

In other words, f is very slowly oscillating if and only if the composition $f \circ \sinh$ is uniformly continuous with respect the standard Euclidean metric on $\mathbb{R}$. The set (C^*-algebra) of all such functions is denoted by $\mathrm{VSO}(\mathbb{R})$.

It is useful to write the spectral function $\gamma_{a,\lambda}$ given in (1.4) as the value of the integral operator

$$\gamma_{a,\lambda}(x) = \int_0^\pi a(\theta) K_\lambda(x,\theta)\, d\theta,$$

where

$$K_\lambda(x,\theta) = \frac{e^{-2x\theta} \sin^\lambda \theta}{c_\lambda(x)}, \quad (x,\theta) \in \mathbb{R} \times (0,\pi). \tag{3.2}$$

Next, we introduce a metric ζ_λ on $\mathbb{R}$ which is the "most natural" for the functions $\gamma_{a,\lambda}$, and show that ζ_λ can be estimated from above by the arcsinh-metric ρ.

Proposition 3.3. *Let $\zeta_\lambda \colon \mathbb{R} \times \mathbb{R} \to [0,+\infty)$ be given by*

$$\zeta_\lambda(x,y) = \sup_{\substack{a \in L_\infty(0,\pi) \\ \|a\|_\infty = 1}} |\gamma_{a,\lambda}(x) - \gamma_{a,\lambda}(y)|. \tag{3.3}$$

Then for every $x, y \in \mathbb{R}$

$$\zeta_\lambda(x,y) = \int_0^\pi |K_\lambda(x,\theta) - K_\lambda(y,\theta)|\, d\theta. \tag{3.4}$$

Proof. Let $x, y \in \mathbb{R}$. Then for every $a \in L_\infty(0,\pi)$ such that $\|a\|_\infty = 1$ we have

$$\left|\gamma_{a,\lambda}(x) - \gamma_{a,\lambda}(y)\right| = \left|\int_0^\pi a(\theta)\,[K_\lambda(x,\theta) - K_\lambda(y,\theta)]\, d\theta\right| \le \int_0^\pi |K_\lambda(x,\theta) - K_\lambda(y,\theta)|\, d\theta.$$

On the other hand, taking $b_0(\theta) = \operatorname{sign}(K_\lambda(x,\theta) - K_\lambda(y,\theta))$ we get

$$\zeta_\lambda(x,y) \ge \left|\gamma_{b_0}(x) - \gamma_{b_0}(y)\right| = \int_0^\pi |K_\lambda(x,\theta) - K_\lambda(y,\theta)|\, d\theta. \qquad \square$$

Let us mention some symmetry properties of K_λ and ζ_λ.

Lemma 3.4. *For every $x \in \mathbb{R}$ and every $\theta \in (0,\pi)$,*

$$K_\lambda(-x,\theta) = K(x,\pi - \theta). \tag{3.5}$$

Proof. First we make the change of variables $\eta = \pi - \theta$ in the integral (1.3) defining c_λ:

$$c_\lambda(-x) = \int_0^\pi e^{2x\theta} \sin^\lambda \theta\, d\theta = e^{2\pi x} \int_0^\pi e^{-2x\eta} \sin^\lambda(\pi - \eta)\, d\eta = e^{2\pi x} c_\lambda(x). \tag{3.6}$$

After that the identity (3.5) is obtained by a direct computation. $\square$

Lemma 3.5. (i) $\zeta_\lambda(x,y) = \zeta_\lambda(-x,-y)$ *for every $x, y \in \mathbb{R}$.*
(ii) *If $x \le 0 \le y$, then $\zeta_\lambda(x,y) \le \zeta_\lambda(x,0) + \zeta_\lambda(0,y)$.*

Proof. (i) follows from Lemma 3.4, (ii) follows from the Triangle Inequality. $\square$

Lemma 3.6. *The function c_λ defined by (1.3) is infinitely smooth; for every $p \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ its p-th derivative is given by*

$$\frac{d^p c_\lambda(x)}{dx^p} = (-2)^p \int_0^\pi \theta^p e^{-2x\theta} \sin^\lambda \theta \, d\theta. \tag{3.7}$$

Moreover, it has the following asymptotic behavior at $+\infty$:

$$\frac{d^p c_\lambda(x)}{dx^p} \sim \frac{(-2)^p \Gamma(\lambda + p + 1)}{(2x)^{\lambda+p+1}}, \quad as \quad x \to +\infty. \tag{3.8}$$

Proof. Due to (3.6) it is sufficient to prove (3.7) only for the case $x \geq 0$. Given $p \in \mathbb{Z}_+$, note that $\theta^p e^{-2x\theta} \sin^\lambda \theta \leq \theta^p \sin^\lambda \theta$ for each $\theta \in (0, \pi)$ and each $x \geq 0$. Thus, by the Leibniz rule c_λ is infinitely smooth and

$$\frac{d^p c_\lambda(x)}{dx^p} = (-2)^p \int_0^\pi \theta^p e^{-2x\theta} \sin^\lambda \theta \, d\theta, \qquad x \in \mathbb{R}.$$

On the other hand, the asymptotic behavior is easily analized using the Watson Lemma (see, for example, [14, Proposition 2.1]). Writing $\theta^p \sin^\lambda \theta$ as $\theta^{\lambda+p} \left(\frac{\sin\theta}{\theta}\right)^\lambda$, where $\left(\frac{\sin\theta}{\theta}\right)^\lambda$ is infinitely smooth near 0, we obtain:

$$\int_0^\pi \theta^p e^{-2x\theta} \sin^\lambda \theta \, d\theta \sim \frac{\Gamma(\lambda + p + 1)}{(2x)^{\lambda+p+1}}, \qquad \text{as } x \to +\infty. \qquad \square$$

Lemma 3.7. *Let $a_0(\theta) = \theta$. The function $\kappa\colon [0, +\infty) \to (0, \infty)$, given by the formula*

$$\kappa(x) = 2\gamma_{a_0,\lambda}(x) \sqrt{x^2 + 1}, \tag{3.9}$$

is continuous and bounded.

Proof. The function κ, being the product of two continuous functions, is obviously continuous. In [19] it was proved that $\gamma_{a,\lambda}(-x) + \gamma_{a,\lambda}(x) = \pi$ for every $a \in L_\infty(0, \pi)$ and every x. In particular, $\gamma_{a_0,\lambda}(0) = \pi/2$ and thus $\kappa(0) = \pi$. Since

$$\kappa(x) = -\frac{c'_\lambda(x) \sqrt{x^2 + 1}}{c_\lambda(x)}, \qquad x \in \mathbb{R},$$

the asymptotic behavior of $\kappa(x)$ as $x \to +\infty$ can be analyzed using Lemma 3.6. By (3.8) with $p = 0$ and $p = 1$ we obtain:

$$c_\lambda(x) \sim \frac{\Gamma(\lambda + 1)}{(2x)^{\lambda+1}}, \qquad c'_\lambda(x) \sim -\frac{2\Gamma(\lambda + 2)}{(2x)^{\lambda+2}}, \qquad \text{as } x \to +\infty.$$

Thus,

$$\lim_{x\to+\infty} \kappa(x) = \lim_{x\to+\infty} \frac{2\Gamma(\lambda + 2)(2x)^{\lambda+1} \sqrt{x^2 + 1}}{\Gamma(\lambda + 1)(2x)^{\lambda+2}} = \lambda + 1,$$

and κ is bounded. $\square$

Lemma 3.8. *If $a_0(\theta) = \theta$, then*

$$\int_0^\pi \left| \frac{\partial K_\lambda(x, \theta)}{\partial x} \right| d\theta \leq 4\gamma_{a_0,\lambda}(x), \qquad x \in \mathbb{R}. \tag{3.10}$$

Proof. An easy computation shows that

$$\frac{\partial K_\lambda(x,\theta)}{\partial x} = 2K_\lambda(x,\theta)\left[\frac{\int_0^\pi \beta\, e^{-2x\beta}\, \sin^\lambda\beta\, d\beta}{\int_0^\pi e^{-2x\beta}\, \sin^\lambda\beta\, d\beta} - \theta\right] = 2K_\lambda(x,\theta)\left[\gamma_{a_0,\lambda}(x) - \theta\right].$$

Therefore

$$\int_0^\pi \left|\frac{\partial K_\lambda(x,\theta)}{\partial x}\right| d\theta \le 2\left(\gamma_{a_0,\lambda}(x)\int_0^\pi K_\lambda(x,\theta)\sin^\lambda\theta\, d\theta + \int_0^\pi \theta\, K_\lambda(x,\theta)\, \sin^\lambda\theta\, d\theta\right)$$
$$= 2\left(\gamma_{a_0,\lambda}(x) + \gamma_{a_0,\lambda}(x)\right) = 4\gamma_{a_0,\lambda}(x).$$
□

Proposition 3.9. *There exists $C > 0$ such that $\zeta_\lambda(x,y) \le C\rho(x,y)$ for every $x,y \in \mathbb{R}$.*

Proof. Due to Lemma 3.5, we only have to consider the case $y > x \ge 0$. By Cauchy's Mean-Value Theorem there exists $c \in (x,y)$ such that

$$\frac{\zeta_\lambda(x,y)}{\rho(x,y)} = \int_0^\pi \left|\frac{K_\lambda(x,\theta) - K_\lambda(y,\theta)}{\operatorname{arcsinh}(y) - \operatorname{arcsinh}(x)}\right| d\theta \le 4\gamma_{a_0,\lambda}(c)\sqrt{c^2+1},$$

and Lemma 3.7 yields the result. □

Theorem 3.10. $\Gamma_\lambda \subsetneq \mathrm{VSO}(\mathbb{R})$.

Proof. Let $a \in L_\infty(0,\pi)$. The inequality $\|\gamma_{a,\lambda}\|_\infty \le \|a\|_\infty$ shows that the function $\gamma_{a,\lambda}$ is bounded, and Proposition 3.9 implies that $\gamma_{a,\lambda}$ is Lipschitz continuous with respect to ρ:

$$|\gamma_{a,\lambda}(x) - \gamma_{a,\lambda}(y)| \le \|a\|_\infty \zeta_\lambda(x,y) \le C\,\|a\|_\infty \rho(x,y), \qquad x,y \in \mathbb{R}.$$

Observe now that the function $\eta(x) = \frac{x^{1/3}}{x^2+1}$ is uniformly continuous, but not Lipschitz on $\mathbb{R}$. Consequently, the composition $\eta \circ \operatorname{arcsinh}$ belongs to $\mathrm{VSO}(\mathbb{R})$, but it is not Lipschitz continuous with respect to ρ and therefore does not belong to Γ_λ. □

4. Density of Γ_λ in $\mathrm{VSO}(\mathbb{R})$

Given $x,y > 0$, by Cauchy's Mean Value Theorem the arcsinh-metric ρ satisfies the inequality

$$\rho(x,y) \le |\ln(x) - \ln(y)|. \tag{4.1}$$

Therefore, if $f \in \mathrm{VSO}(\mathbb{R})$, then $f|_{\mathbb{R}_+} \in \mathrm{VSO}(\mathbb{R}_+)$, where the class of functions $\mathrm{VSO}(\mathbb{R}_+)$ was defined in [12] and mentioned in the Introduction. Furthermore, Herrera Yañez, Hutník, and Maximenko [11] have shown that for every $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$ and $\varepsilon > 0$ there exists $b \in L_\infty(\mathbb{R}_+)$ such that

$$\sup_{x\in\mathbb{R}_+} \left|\sigma(x) - \gamma^{\nu}_{b,\lambda}(x)\right| < \varepsilon,$$

where

$$\gamma^{\nu}_{b,\lambda}(x) = \frac{(2x)^{\lambda+1}}{\Gamma(\lambda+1)}\int_0^\infty b(t)e^{-2xt}t^\lambda\, dt, \qquad x \in \mathbb{R}_+. \tag{4.2}$$

The above considerations lead to the following lemma.

Lemma 4.1. *Let $f \in \mathrm{VSO}(\mathbb{R})$. Given $\varepsilon > 0$ there exists $b \in L_\infty(\mathbb{R}_+)$ such that*

$$\sup_{x\in\mathbb{R}_+} \left|f(x) - \gamma^{\nu}_{b,\lambda}(x)\right| < \varepsilon. \tag{4.3}$$

Lemma 4.2 (approximation of γ by γ^ν at $+\infty$). *If $b \in L_\infty(\mathbb{R}_+)$ and $a = \chi_{(0,\pi/2)}b$, then*

$$\lim_{x\to+\infty} \left|\gamma^\nu_{b,\lambda}(x) - \gamma_{a,\lambda}(x)\right| = 0. \tag{4.4}$$

Proof. Given $x \geq 0$, we have

$$\begin{aligned}\left|\gamma^\nu_{b,\lambda}(x) - \gamma_{a,\lambda}(x)\right| &\leq \|b\|_\infty \int_0^\infty \left|\chi_{(0,\pi/2)}K_\lambda(x,\theta) - \frac{(2x)^{\lambda+1}\,\theta^\lambda\, e^{-2x\theta}}{\Gamma(\lambda+1)}\right| d\theta \\ &\leq \|b\|_\infty \left(\mathrm{I}_1(x) + \mathrm{I}_2(x)\right),\end{aligned}$$

where

$$\begin{aligned}\mathrm{I}_1(x) &= \int_0^{\pi/2} \left|K_\lambda(x,\theta) - \frac{(2x)^{\lambda+1}\, e^{-2x\theta}\, \sin^\lambda\theta}{\Gamma(\lambda+1)}\right| d\theta, \\ \mathrm{I}_2(x) &= \frac{(2x)^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty e^{-2x\theta}\left|\theta^\lambda - \chi_{(0,\pi/2)}\sin^\lambda\theta\right| d\theta.\end{aligned}$$

By (3.2),

$$\mathrm{I}_1(x) \leq \int_0^\pi \left|\frac{e^{-2x\theta}\sin^\lambda\theta}{c_\lambda(x)} - \frac{(2x)^{\lambda+1}\, e^{-2x\theta}\sin^\lambda\theta}{\Gamma(\lambda+1)}\right| d\theta = \left|1 - \frac{(2x)^{\lambda+1}c_\lambda(x)}{\Gamma(\lambda+1)}\right|,$$

where c_λ is given in (1.3). By (3.8) with $p = 0$, we obtain $\lim_{x\to+\infty} \mathrm{I}_1(x) = 0$.

On the other hand, the integral I_2 can be written as

$$\mathrm{I}_2(x) = \frac{(2x)^{\lambda+1}}{\Gamma(\lambda+1)}\left(\int_0^{\pi/2} e^{-2x\theta}\theta^\lambda \left|\left(\frac{\sin\theta}{\theta}\right)^\lambda - 1\right| d\theta\right) + \frac{\Gamma(\lambda+1, x\pi)}{\Gamma(\lambda+1)},$$

where $\Gamma(\alpha, x)$ is the incomplete Gamma function. We see for every $\theta \in (0, \pi/2)$ that the function

$$\left|\left(\frac{\sin\theta}{\theta}\right)^\lambda - 1\right| = \begin{cases}\left(\frac{\sin\theta}{\theta}\right)^\lambda - 1, & \text{if } \lambda \geq 0 \\ 1 - \left(\frac{\sin\theta}{\theta}\right)^\lambda, & \text{if } -1 < \lambda \leq 0\end{cases}$$

is infinitely smooth near 0 and vanishes in 0. Then by Watson Lemma and the definition of $\Gamma(\alpha, x)$ we get $\lim\limits_{x\to+\infty} \mathrm{I}_2(x) = 0$, which yields (4.4). □

The above lemmas permit us now to show that each $f \in \mathrm{VSO}(\mathbb{R})$ can be approximated by functions from the class Γ_λ for large values of $|x|$.

Proposition 4.3. *Let $f \in \mathrm{VSO}(\mathbb{R})$ and $\varepsilon > 0$. Then there exist a generating symbol $a \in L_\infty(0,\pi)$ and a number $L > 0$ such that*

$$\sup_{|x|\geq L} \left|f(x) - \gamma_{a,\lambda}(x)\right| \leq \varepsilon. \tag{4.5}$$

Proof. Given $f \in \mathrm{VSO}(\mathbb{R})$ and $\varepsilon > 0$ there exist $b \in L_\infty(\mathbb{R}_+)$ such that (4.3) holds. By Lemma 4.2 there exist $c \in L_\infty(0,\pi)$ with $c(\theta) = 0$ for each $\theta \in [\pi/2, \pi)$, and $L_1 > 0$ such that

$$\sup_{x\geq L_1} \left|f(x) - \gamma_{c,\lambda}(x)\right| \leq \sup_{x\geq L_1}\left(\left|f(x) - \gamma^\nu_{b,\lambda}(x)\right| + \left|\gamma^\nu_{b,\lambda}(x) - \gamma_{c,\lambda}(x)\right|\right) \leq \varepsilon. \tag{4.6}$$

For large negative values of x, we consider the function $x \mapsto f(-x)$ that also belongs to $\mathrm{VSO}(\mathbb{R})$. Applying the previous arguments to this function we find a function $g \in L_\infty(0,\pi)$ and a number $L_2 > 0$ such that g vanishes in $[\pi/2, \pi)$ and

$$\sup_{x \geq L_2} |f(-x) - \gamma_{g,\lambda}(x)| \leq \varepsilon. \tag{4.7}$$

Now define $d \in L_\infty(0,\pi)$ by $d(\theta) = g(\pi - \theta)$. Then d vanishes on $(0, \pi/2]$, and the identity $\gamma_{d,\lambda}(x) = \gamma_{g,\lambda}(-x)$ holds. Hence (4.7) can be rewritten as

$$\sup_{x \leq -L_2} |f(x) - \gamma_{d,\lambda}(x)| \leq \varepsilon. \tag{4.8}$$

Since c vanishes near π and d vanishes near 0, the corresponding spectral functions fulfill the limit relations $\gamma_{c,\lambda}(-\infty) = 0$ and $\gamma_{d,\lambda}(+\infty) = 0$ (see [19, Chapter 14]), and there are constants $L_3, L_4 > 0$ such that

$$\sup_{x \leq -L_3} |\gamma_{c,\lambda}(x)| \leq \frac{\varepsilon}{2}, \qquad \sup_{x \geq L_4} |\gamma_{d,\lambda}(x)| \leq \frac{\varepsilon}{2}.$$

Taking $a = d + c \in L_\infty(0,\pi)$ and $L = \max\{L_1, L_2, L_3, L_4\}$, we get (4.4). □

We show now that the continuous functions on $\mathbb{R}$ vanishing at the infinity can be approximated by spectral functions. To do that, we need some technical lemmas.

Lemma 4.4. *Let $(\Omega, \mathcal{A})$ be a measurable space, $\mu\colon \mathcal{A} \to \mathbb{C}$ be a complex measure, D be a domain in $\mathbb{C}$ and $K\colon D \times \Omega \to \mathbb{C}$ be a function such that for every $\omega \in \Omega$ the function $z \mapsto K(z, \omega)$ is analytic, and for every compact C in D*

$$\sup_{z \in C} \int_\Omega |K(z, \omega)|\, d|\mu|(\omega) < +\infty.$$

Then the function $g\colon D \to \mathbb{C}$,

$$g(z) = \int_\Omega K(z, \omega)\, d\mu(\omega),$$

is analytic on D.

Proof. The proof is immediate by Fubini and Morera theorems. □

From now on, we write K_λ as

$$K_\lambda(x, \theta) = F_\lambda(x, \theta) \sin^\lambda \theta,$$

with

$$F_\lambda(x, \theta) = \frac{e^{-2x\theta}}{c_\lambda(x)}, \quad (x, \theta) \in \mathbb{R} \times (0, \pi). \tag{4.9}$$

Lemma 4.5. *Let $p \in \mathbb{Z}_+$, and let ν be a finite regular Borel complex measure on $\mathbb{R}$, and $\delta \in (0, \pi/2)$. Then*

$$\sup_{\delta \leq \alpha \leq \pi - \delta} \int_{\mathbb{R}} |x^p F_\lambda(x, \alpha)|\, d|\nu|(x) < +\infty. \tag{4.10}$$

Proof. By (3.8) the function $F_\lambda : \mathbb{R} \times (0, \pi) \to (0, +\infty)$ given in (4.9) has an asymptotic behavior

$$F_\lambda(x, \theta) \sim \frac{(2x)^{\lambda+1} e^{-2x\theta}}{\Gamma(\lambda+1)}, \quad \text{as } x \to +\infty. \tag{4.11}$$

Hence, given $p \in \{0, 1, 2, \ldots, \}$, $\alpha \in [\delta, \pi - \delta]$ with $\delta \in (0, \pi/2)$ and $x \in [0, +\infty)$, we get

$$|x^p F_\lambda(x, \alpha)| \sim \frac{(2x)^{\lambda+1+p} e^{-2x\alpha}}{2^p \Gamma(\lambda+1)} \le \frac{(2x)^{\lambda+1+p} e^{-2x\delta}}{2^p \Gamma(\lambda+1)} \le M_{\lambda,p,\delta} \quad \text{as } x \to +\infty.$$

However, from Lemma 3.4 we have $F_\lambda(x, \alpha) = F_\lambda(-x, \beta)$, where $\beta = \pi - \alpha \in [\delta, \pi - \delta]$. Therefore $F_\lambda(x, \alpha)$ is bounded for all $\alpha \in [\delta, \pi - \delta]$. □

Proposition 4.6 (Leibniz integral rule for differentiation under the integral sign: complex case). *Let X be an open subset of $\mathbb{R}$, $(\Omega, \mathcal{A})$ be a measurable space and $\mu\colon \mathcal{A} \to \mathbb{C}$ be a complex measure. Suppose $f\colon X \times \Omega \to \mathbb{R}$ satisfies the following conditions:*

(i) *For every $x \in X$, the function $\omega \mapsto f(x, \omega)$ is $|\mu|$-integrable.*
(ii) *For almost all ω in Ω, the derivative f_x exists for all x in X.*
(iii) *There is a $|\mu|$-integrable function $\theta\colon \Omega \to \mathbb{R}$ such that $|f_x(x, \omega)| \le \theta(\omega)$ for all $x \in X$.*

Then for all x in X

$$\frac{d}{dx} \int_\Omega f(x, \omega)\, d\mu(\omega) = \int_\Omega f_x(x, \omega)\, d\mu(\omega).$$

Proof. The Leibniz rule is well known in the case of a non-negative measure, but every complex measure μ can be written as a linear combination of four non-negative measures $\mu_1, \mu_2, \mu_3, \mu_4$, with $\mu_j \le |\mu|$. The conditions (i) and (iii) justify the application of the Leibniz rule for each one of the measures μ_j. □

Lemma 4.7. *Let ν be a regular complex Borel measure of a finite total variation on $\mathbb{R}$. Define a function $\psi_\lambda\colon \mathbb{R} \to \mathbb{C}$ by*

$$\psi_\lambda(x) = F_\lambda(x, \pi/2), \qquad x \in \mathbb{R}. \tag{4.12}$$

Let $\Delta = \{w \in \mathbb{C}\colon\ |\operatorname{Im} w| < \pi\}$, and define $\Phi_\lambda\colon \Delta \to \mathbb{C}$ by

$$\Phi_\lambda(w) = \int_\mathbb{R} e^{-ixw} \psi_\lambda(x)\, d\nu(x). \tag{4.13}$$

Then Φ_λ is analytic on Δ and for every $p \in \mathbb{Z}_+$

$$\Phi_\lambda^{(p)}(0) = (-i)^p \int_\mathbb{R} x^p \psi_\lambda(x)\, d\nu(x). \tag{4.14}$$

Proof. By Lemma 3.4, ψ_λ is an even function: $\psi_\lambda(-x) = \psi_\lambda(x)$.

Every compact subset of Δ is contained in a strip of the form $\mathbb{R} + i[-L, L]$, where $0 < L < \pi$. For every $w \in \mathbb{C}$ with $|\operatorname{Im} w| \le L$ and every $x \in \mathbb{R}$,

$$|e^{-ixw} \psi_\lambda(x)| = e^{x \operatorname{Im}(w)}\, \psi_\lambda(|x|) \le e^{|x| L} \psi_\lambda(|x|) = \frac{1}{e^{|x|(\pi - L)} c_\lambda(|x|)}.$$

The condition $\pi - L > 0$ and Lemma 3.6 guarantee that the latter expression defines a bounded function on $\mathbb{R}$.

Since the complex measure ν has a finite total variation, Lemma 4.4 assures that Φ_λ is analytic in Δ. Thus, by (4.10) and the Leibniz rule (Proposition 4.6) we get

$$\Phi_\lambda^p(z) = \frac{d^p}{dz^p}\left(\int_{\mathbb{R}} \psi_\lambda(x)e^{-ixz}\,d\nu(x)\right) = (-i)^p \int_{\mathbb{R}} x^p\psi_\lambda(x)e^{-ixz}\,d\nu(x). \qquad \square$$

Theorem 4.8. *The set of functions*

$$\Gamma^\lambda_{(0,\pi)} = \{\gamma_{a,\lambda} : a \in C_0(0,\pi)\} \tag{4.15}$$

is a dense subset of $C_0(\mathbb{R})$.

Proof. First we note that if $a \in C_0(0,\pi)$, then $\lim\limits_{x\to+\infty} \gamma_{a,\lambda}(x) = \lim\limits_{\theta\to 0} a(\theta) = 0$ and $\lim_{x\to-\infty} \gamma_{a,\lambda}(x) = \lim_{\theta\to\pi} a(\theta) = 0$ (see [19, Section 14.1]), thus $\Gamma^\lambda_{(0,\pi)} \subseteq C_0(\mathbb{R})$.

By Hahn-Banach theorem, the density of $\Gamma^\lambda_{(0,\pi)}$ in $C_0(0,\pi)$ will be shown if we prove that any continuous linear functional φ on $C_0(0,\pi)$ that vanishes on $\Gamma^\lambda_{(0,\pi)}$ is the zero functional. Thus, let $\varphi \in C_0(\mathbb{R})^*$ be a linear functional such that $\varphi(\gamma_{a,\lambda}) = 0$ for each $a \in L_\infty(0,\pi)$. By Riesz-Markov representation theorem, there is a regular complex Borel measure ν of finite total variation on $\mathbb{R}$ such that

$$0 = \varphi(\gamma_{a,\lambda}) = \int_{\mathbb{R}} \gamma_{a,\lambda}(x)\,d\nu(x), \qquad a \in L_\infty(0,\pi).$$

In particular, if $a_0 = \chi_{[\beta,\theta]}$ with $0 < \beta < \theta < \pi$, then by

$$\int_{\mathbb{R}}\int_0^\pi |a_0(\theta)K_\lambda(x,\theta)|\,d\theta\,d|\nu|(x) \le \int_{\mathbb{R}}\int_0^\pi K_\lambda(x,\theta)\,d\theta\,d|\nu|(x) = |\nu|(\mathbb{R}),$$

we can apply Fubini's theorem and get

$$\int_{\mathbb{R}} \gamma_{a_0,\lambda}(x)\,d\nu(x) = \int_{\mathbb{R}}\int_\beta^\theta K_\lambda(x,\alpha)\,d\alpha\,d\nu(x) = \int_\beta^\theta\int_{\mathbb{R}} K_\lambda(x,\alpha)\,d\nu(x)\,d\alpha = 0.$$

The function $\alpha \to \int_{\mathbb{R}} K_\lambda(x,\alpha)\,d\nu(x)$ is continuous (in fact, it is differentiable, see below), therefore by the first fundamental theorem of calculus we obtain that for every θ in $(0,\pi)$

$$\int_{\mathbb{R}} K_\lambda(x,\theta)\,d\nu(x) = 0.$$

Since $K_\lambda(x,\theta) = F_\lambda(x,\theta)\sin^\lambda\theta$ and $\sin\theta > 0$, this is equivalent to

$$\int_{\mathbb{R}} F_\lambda(x,\theta)\,d\nu(x) = 0. \tag{4.16}$$

By Lemma 4.5 and Leibniz rule, the function in the left-hand side of (4.16) is differentiable, and the derivation with respect to θ commutes with the integral sign. Derivating (4.16) with respect to θ we obtain for every θ in $(0,\pi)$ and every p in $\mathbb{Z}_+$

$$\int_{\mathbb{R}} x^p F_\lambda(x,\theta)\,d\nu(x) = 0. \tag{4.17}$$

Choosing $\theta = \pi/2$ in (4.17) we obtain for every p in $\mathbb{Z}_+$:

$$\int_{\mathbb{R}} x^p\,\psi_\lambda(x)\,d\nu(x) = 0, \tag{4.18}$$

where $\psi_\lambda\colon \mathbb{R} \to \mathbb{C}$ is given by $\psi_\lambda(x) = F_\lambda(x, \pi/2)$. Denote by Φ_λ the Fourier transform of the measure $\psi_\lambda\, d\nu$:

$$\Phi_\lambda(\xi) = \int_{\mathbb{R}} e^{-ix\xi} \psi_\lambda(x)\, d\nu(x).$$

Lemma 4.7 shows that the function Φ_λ is analytic on a domain containing $\mathbb{R}$, and (4.18) means that $\Phi_\lambda^{(p)}(0) = 0$ for every $p \in \mathbb{Z}_+$. Therefore $\Phi_\lambda = 0$. By the injective property of the Fourier transform of Borel measures (see, for example, [5, Proposition 3.8.6]), we conclude that $\nu = 0$ and hence $\varphi = 0$. That implies the density of $\Gamma^\lambda_{(0,\pi)}$ in $C_0(\mathbb{R})$. □

Proposition 4.3 and Theorem 4.8 imply together the main result on density.

Theorem 4.9. *The set* Γ_λ *is dense in* VSO($\mathbb{R}$).

Proof. Let $f \in \mathrm{VSO}(\mathbb{R})$ and $\varepsilon > 0$. Our aim is to find a function c in $L_\infty(0, \pi)$ such that $\|f - \gamma_{c,\lambda}\|_\infty \leq \varepsilon$. First, using Lemma 4.3 we find a function $a \in L_\infty(0, \pi)$ and a number $L > 0$ such that $\sup_{|x|\geq L} \left|f(x) - \gamma_{a,\lambda}(x)\right| \leq \frac{\varepsilon}{2}$. In general, the function $f - \gamma_{a,\lambda}$ may not belong to the class $C_0(\mathbb{R})$, and we will slightly modify it. Let $g\colon \mathbb{R} \to [0, 1]$ be a continuous function such that $g(x) = 1$ for each $x \in [-2L, 2L]$ and $g(x) = 0$ for each $x \in \mathbb{R} \setminus [-2L, 2L]$. Define $h \in C_0(\mathbb{R})$ by

$$h(x) = (f - \gamma_{a,\lambda})(x)g(x) = \begin{cases} f(x) - \gamma_{a,\lambda}(x), & \text{if } |x| \leq L; \\ (f(x) - \gamma_{a,\lambda}(x))g(x), & \text{if } L < |x| \leq 2L; \\ 0, & \text{if } |x| > 2L. \end{cases}$$

Second, applying Theorem 4.8 we choose $b \in L_\infty(0, \pi)$ such that $\|h - \gamma_{b,\lambda}\|_\infty \leq \varepsilon/2$. Now define $c \in L_\infty(0, \pi)$ by $c = a + b$. Then for every x in $[-L, L]$ we obtain

$$|f(x) - \gamma_{c,\lambda}(x)| = |f(x) - \gamma_{a,\lambda}(x) - \gamma_{b,\lambda}(x)| = |h(x) - \gamma_{b,\lambda}(x)| \leq \varepsilon/2,$$

and for every x in $\mathbb{R} \setminus [-L, L]$

$$\begin{aligned} |f(x) - \gamma_{c,\lambda}(x)| &= |(f(x) - \gamma_{a,\lambda}(x))(1 - g(x)) + (f(x) - \gamma_{a,\lambda}(x))g(x) - \gamma_{b,\lambda}(x)| \\ &\leq |f(x) - \gamma_{a,\lambda}(x)|\,(1 - g(x)) + |h(x) - \gamma_{b,\lambda}(x)| \leq \varepsilon. \end{aligned}$$

Therefore $\|f - \gamma_{c,\lambda}\|_\infty \leq \varepsilon$. □

Corollary 4.10. *The* C^**-algebra generated by* Γ_λ *coincides with* VSO($\mathbb{R}$)*, and the* C^**-algebra* $\mathcal{T}_\lambda(\mathcal{A}_\infty)$ *generated by angular Toeplitz operators is isometrically isomorphic to* VSO($\mathbb{R}$).

Huang [13] proved that if $T \in \mathcal{B}(L_2(\mathbb{R}))$ commutes with the multiplication operator M_φ, where φ is a bounded strictly increasing (or decreasing) function on $\mathbb{R}$, then $T = M_\psi$, for some $\psi \in L_\infty(\mathbb{R})$. Now, since each angular Toeplitz operator T_a is unitarily equivalent to the multiplication operator M_{γ_a}, the above Huang result implies that the von Neumann algebra $W^*(\mathrm{T}_\lambda(\mathcal{A}_\infty))$ generated by $\mathrm{T}_\lambda(\mathcal{A}_\infty)$ is maximal. In fact, $W^*(\mathrm{T}_\lambda(\mathcal{A}_\infty))$ is the closure of $\mathcal{T}_\lambda(\mathcal{A}_\infty)$ with respect to the strong operator topology (SOT) in $\mathcal{B}(\mathcal{A}^2_\lambda(\Pi))$.

The space $L_\infty(\mathbb{R})$ may be identified with the dual space of $L_1(\mathbb{R})$. We denote by $\mathcal{W}$ the corresponding weak-* topology on $L_\infty(\mathbb{R})$. The continuity of Lebesgue integral, Lusin theorem and Urysohn Lemma prove that $C_0(\mathbb{R})$ is dense in $(L_\infty(\mathbb{R}), \mathcal{W})$.

Using this fact and the main density result, we complement the above Huang result providing an explicit description of the SOT-closure of $\mathrm{T}_\lambda(\mathcal{A}_\infty)$.

The following proposition states the density of $\mathrm{T}_\lambda(\mathcal{A}_\infty)$ in the C^*-algebra $\mathfrak{A}_\lambda$ with respect to the strong operator topology in $\mathcal{B}(\mathcal{A}^2_\lambda(\Pi))$.

Proposition 4.11. SOT-closure$(\mathrm{T}_\lambda(\mathcal{A}_\infty)) = \mathfrak{A}_\lambda$.

Proof. Since $C_0(\mathbb{R})$ is dense in $(L_\infty(\mathbb{R}), \mathcal{W})$, by Theorem 4.8 (or Theorem 4.9) we deduce that Γ_λ is dense in $(L_\infty(\mathbb{R}), \mathcal{W})$. Now, due to a net in $(L_\infty(\mathbb{R}), \mathcal{W})$ converges if and only if its respective multiplication operator in $\mathcal{B}(L_2(\mathbb{R}))$ converges in the weak operator topology (see for example [6, Proposition 10.5]) we conclude that WOT-closure $(\mathcal{T}_\lambda(\mathcal{A}_\infty)) = \mathfrak{A}_\lambda$. But $\mathrm{T}_\lambda(\mathcal{A}_\infty)$ is a convex subset of $\mathcal{B}(\mathcal{A}^2_\lambda(\Pi))$, thus the SOT-closure and the WOT-closure coincide. □

Example 4.1. We present an example of a function $\gamma_{a,\lambda}$ that has a typical "very slow oscillation" at $\pm\infty$. Consider the generating symbol

$$a(\theta) = \cos(\ln(\tan(\theta/2))).$$

Then $a(\pi - \theta) = a(\theta)$ and $\gamma_{a,\lambda}(-x) = \gamma_{a,\lambda}(x)$. Watson lemma implies that the asymptotical behavior of $\gamma_{a,\lambda}(x)$ as $x \to +\infty$ is determined by the behaviour of a near the point 0, and $\tan(\theta/2) \sim \theta/2$ as $\theta \to 0$. Using arguments similar to those in the proof of Lemma 4.2 we see that as $x \to +\infty$,

$$\gamma_{a,\lambda}(x) = \frac{1}{c_\lambda(x)} \int_0^{+\infty} \theta^\lambda e^{-2x\theta} \cos(\ln(\theta/2))\, d\theta + o(1) = \frac{\mathrm{Re}((2x)^i\, \Gamma(1 - i + \lambda))}{\Gamma(\lambda + 1)} + o(1).$$

With the change of variables $x = \sinh(u)$ and applying the limit relation $|\sinh(u)| \sim \exp(|u|)/2$ we obtain that

$$\gamma_{a,\lambda}(\sinh(u)) = \frac{|\Gamma(1 - i + \lambda)|}{\Gamma(\lambda + 1)} \cos(|u| + \ln 2 + \arg\Gamma(1 - i + \lambda)) + o(1),$$

as $u \to \pm\infty$.

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