Vertical Toeplitz operators on the upper half-plane and very slowly oscillating functions

Crispin Herrera Yañez, Egor A. Maximenko, Nikolai Vasilevski

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Abstract

We consider the C^*-algebra generated by Toeplitz operators acting on the Bergman space over the upper half-plane whose symbols depend on the imaginary part of the argument only. Such algebra is known to be commutative, and is isometrically isomorphic to an algebra of bounded complex-valued functions on the positive half-line. In the paper we prove that the latter algebra consists of all bounded functions f that are very slowly oscillating in the sense that the composition of f with the exponential function is uniformly continuous or, in other words,

$$\lim_{\frac{x}{y}\to 1}\big|f(x)-f(y)\big|=0.$$

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1 Introduction

The paper is devoted to the description of a certain class of Toeplitz operators acting on the Bergman space over the upper half-plane and of the C^*-algebra generated by them.

Let $\Pi = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane, and let $d\mu = dxdy$ be the standard Lebesgue plane measure on Π. Recall that the Bergman space $\mathcal{A}^2(\Pi)$ is the (closed) subspace of $L_2(\Pi, d\mu)$ which consists of all function analytic in Π. It is well known that $\mathcal{A}^2(\Pi)$ is a reproducing kernel Hilbert space whose (Bergman) reproducing kernel has the form

$$K_{\Pi,w}(z) = -\frac{1}{\pi(\overline{w} - z)^2};$$

thus the Bergman (orthogonal) projection of $L_2(\Pi, d\mu)$ onto $\mathcal{A}^2(\Pi)$ is given by

$$(Pf)(w) = \langle f, K_{\Pi,w} \rangle.$$

Given a function $g \in L_\infty(\Pi)$, the Toeplitz operator $T_g \colon \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ with generating symbol g is defined by $T_g f = P(gf)$.

One of the phenomena in the theory of Toeplitz operators on the Bergman space is that (contrary to the Hardy space case) there exists a rich family of symbols that generate *commutative* algebras of Toeplitz operators (see for details [13, 14]). There are three model classes of such symbols: *elliptic*, which is realized by radial symbols, functions depending on $|z|$, on the unit disk, *parabolic*, which is realized by symbols depending on $y = \mathrm{Im}(z)$ on the upper half-plane, and *hyperbolic*, which is realized by homogeneous of order zero symbols on the upper half-plane. All other classes of symbols, that generate commutative algebras of Toeplitz operators, are obtained from the above three model classes by means of the Möbius transformations.

In each case of a commutative algebra of Toeplitz operators there is an (explicitly defined) unitary operator R that reduces each Toeplitz operator T_a from the algebra to a certain (again explicitly given) multiplication operator by γ_a, being a function (in the parabolic and hyperbolic cases), or a sequence (in the elliptic case). This "spectral" function (or sequence) γ_a "carries" many substantial properties of corresponding Toeplitz operators, such as boundedness, norm, compactness, spectrum, essential spectrum, etc.
A very important task to be done in this connection is to describe the properties of such "spectral" functions and algebras generated by them, understanding thus in more detail the properties of Toeplitz operators and algebras generated by them.

The first essential step in this direction was done by Suárez [9, 10], who proved, in particular, that the set of Toeplitz operators with bounded radial symbols (the elliptic case) is dense in the C^*-algebra generated by these operators, and that the l_∞-closure of the set of corresponding "spectral" sequences coincides with the l_∞-closure of a certain set, which he denotes by d_1 and which is commonly used in Tauberian theory. Then in [4] it was shown that this closure coincides with the C^*-algebra of all *slowly oscillating sequences* introduced by Schmidt [7, Definition 10], i.e., of all bounded sequences $x = (x_n)_{n=0}^{\infty}$ such that

$$\lim_{\frac{m+1}{n+1}\to 1} |x_m - x_n| = 0,$$

which gives thus an isometric characterization of the elliptic case commutative algebra.

In this paper we study the commutative C^*-algebra $\mathcal{VT}(L_\infty)$ generated by Toeplitz operators of the model parabolic case, i.e., by Toeplitz operators with bounded symbols depending on $y = \mathrm{Im}(z)$ (we call such symbols *vertical*). The main result of the paper states that the set of Toeplitz operators with bounded vertical symbols is dense in the above C^*-algebra, and that the algebra $\mathcal{VT}(L_\infty)$ itself is isometrically isomorphic to the (introduced in the paper) C^*-algebra $\mathrm{VSO}(\mathbb{R}_+)$ of *very slowly oscillating functions*, the functions that are uniformly continuous with respect to the *logarithmic metric* $\rho(x, y) = |\ln(x) - \ln(y)|$ on $\mathbb{R}_+$ or, equivalently, the functions satisfying the condition

$$\lim_{\frac{x}{y}\to 1} \bigl|f(x) - f(y)\bigr| = 0.$$

The paper is organized as follows. In Sections 2 and 3 we give various equivalent descriptions of *vertical* operators (operators that are invariant under horizontal shifts) and of vertical Toeplitz operators. In Sections 4 and 5 we introduce the algebra $\mathrm{VSO}(\mathbb{R}_+)$ and prove the above stated main result on density. In Section 6 we give an example of a *bounded* Toeplitz operator T_a with *unbounded* vertical symbol a whose "spectral" function γ_a does not belong to the algebra $\mathrm{VSO}(\mathbb{R}_+)$. This means that in spite of its boundedness T_a *does not belong* to the C^*-algebra generated by Toeplitz operators with bounded vertical symbols. In other words, admitting bounded Toeplitz operators with unbounded symbol we enlarge the algebra $\mathcal{VT}(L_\infty)$.

Note that the technique used in the paper for the parabolic case is more simple and efficient than the general one of [9, 10]. Instead of the n-Berezin transform (a special kind of an approximative unit introduced and used by Suárez), we use another approximative unit based on a certain Dirac sequence.

2 Vertical operators

Let $\mathcal{L}(\mathcal{A}^2(\Pi))$ be the algebra of all linear bounded operators acting on the Bergman space $\mathcal{A}^2(\Pi)$. Given $h \in \mathbb{R}$, let $H_h \in \mathcal{L}(\mathcal{A}^2(\Pi))$ be the *horizontal translation operator* defined by

$$H_h f(z) := f(z - h).$$

We call an operator $S \in \mathcal{L}(\mathcal{A}^2(\Pi))$ *vertical* (or *horizontal translation invariant*) if it commutes with all horizontal translation operators:

$$\forall h \in \mathbb{R}, \qquad H_h S = S H_h.$$

In this section we find a criterion for an operator from $\mathcal{A}^2(\Pi)$ to be vertical. First we recall some known facts on translation invariant operators on the real line.

Introduce the standard Fourier transform

$$(Ff)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-ist} f(t)\, dt,$$

being a unitary operator on $L_2(\mathbb{R})$.

For each $h \in \mathbb{R}$, the translation operator $\tau_h \colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is defined by

$$\tau_h f(s) := f(s-h).$$

An operator S on $L_2(\mathbb{R})$ is called *translation invariant* if $\tau_h S = S\tau_h$, for all $h \in \mathbb{R}$. It is well known (see, for example, [5, Theorem 2.5.10]) that an operator S on $L_2(\mathbb{R})$ is translation invariant if and only if it is a convolution operator, i.e., if and only if there exists a function $\sigma \in L_\infty(\mathbb{R})$ such that

$$S = F^{-1} M_\sigma F. \tag{2.1}$$

We introduce as well the associated *multiplication by a character operator* $M_{\Theta_h} f(s) := \Theta_h(s) f(s)$, where $\Theta_h(s) := \mathrm{e}^{ish}$.

Note that τ_h and $M_{\Theta_{-h}}$ are related via the Fourier transform,

$$M_{\Theta_{-h}} F = F\tau_h. \tag{2.2}$$

Lemma 2.1. *Let $M \in \mathcal{L}(L_2(\mathbb{R}))$. The following conditions are equivalent:*

(a) M is invariant under multiplication by Θ_h for all $h \in \mathbb{R}$:

$$M M_{\Theta_h} = M_{\Theta_h} M.$$

(b) M is the multiplication operator by a bounded measurable function:

$$\exists\, \sigma \in L_\infty(\mathbb{R}) \quad \text{such that} \quad M = M_\sigma.$$

Proof. The part (b)⇒(a) is trivial: $M_\sigma M_{\Theta_h} = M_{\sigma\Theta_h} = M_{\Theta_h} M_\sigma$. The implication (a)⇒(b) follows from the relation (2.2) and the result about the translation invariant operators cited above. □

Old proof. Assuming (a), by (2.2) we have

$$F^{-1} M F \tau_h = F^{-1} M M_{\Theta_{-h}} F = F^{-1} M_{\Theta_{-h}} M F = \tau_h F^{-1} M F,$$

which implies that $F^{-1}MF$ commutes with translations. Then (2.1) implies

$$F^{-1} M F = F^{-1} M_\sigma F.$$

Since F is unitary, (b) holds.

Conversely, if (b) holds, then $M_\sigma M_{\Theta_h} = M_{\sigma\Theta_h} = M_{\Theta_h} M_\sigma$. □

Let Θ_h^+ denote the restriction of Θ_h to $\mathbb{R}_+$. The following lemma states that an operator on $L_2(\mathbb{R}_+)$ commutes with $M_{\Theta_h^+}$ if and only if it is a multiplication operator.

Lemma 2.2. *Let $M \in \mathcal{L}(L_2(\mathbb{R}_+))$. The following conditions are equivalent:*

(a) M is invariant under multiplication by Θ_h^+ for all $h \in \mathbb{R}$:

$$MM_{\Theta_h^+} = M_{\Theta_h^+}M.$$

(b) M is the multiplication operator by a bounded function:

$$\exists\, \sigma \in L_\infty(\mathbb{R}_+) \quad \text{such that} \quad M = M_\sigma.$$

Proof. To prove that (a) implies (b), define the *restriction* operator

$$P\colon L_2(\mathbb{R}) \to L_2(\mathbb{R}_+), \qquad g \mapsto g|_{\mathbb{R}_+},$$

and the *zero extension* operator

$$J\colon L_2(\mathbb{R}_+) \to L_2(\mathbb{R}), \qquad Jf(x) := \begin{cases} f(x) & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

For every $h \in \mathbb{R}$ the following equalities hold:

$$JM_{\Theta_h^+} = M_{\Theta_h}J, \qquad PM_{\Theta_h} = M_{\Theta_h^+}P.$$

If (a) holds, then the operator JMP is invariant under multiplication by Θ_h, for all $h \in \mathbb{R}$:

$$JMPM_{\Theta_h} = JMM_{\Theta_h^+}P = JM_{\Theta_h^+}MP = M_{\Theta_h}JMP,$$

and by Lemma 2.2 there exists a function $\sigma_1 \in L_\infty(\mathbb{R})$ such that $JMP = M_{\sigma_1}$. Set $\sigma = \sigma_1|_{\mathbb{R}_+}$. Then for all $f \in L_2(\mathbb{R}_+)$ and all $x \in \mathbb{R}_+$,

$$\begin{aligned}(M_\sigma f)(x) &= \sigma(x)f(x) = \sigma_1(x)(Jf)(x) = (M_{\sigma_1}Jf)(x) \\ &= (JMPJf)(x) = (JMf)(x) = (Mf)(x),\end{aligned}$$

and (b) holds. The implication (b)⇒(a) follows directly, as in the previous lemma. □

The *Berezin transform* [1, 2] of an operator $S \in \mathcal{L}(\mathcal{A}^2(\Pi))$ is the function $\Pi \to \mathbb{C}$ defined by

$$\mathcal{B}(S)(w) := \frac{\langle SK_{\Pi,w}, K_{\Pi,w}\rangle}{\langle K_{\Pi,w}, K_{\Pi,w}\rangle}.$$

Following [12, Section 2] (see also [14, Section 3.1]), we introduce the isometric isomorphism $R\colon \mathcal{A}^2(\Pi) \to L_2(\mathbb{R}_+)$,

$$(R\phi)(x) := \frac{\sqrt{x}}{\sqrt{\pi}} \int_{\Pi} \phi(w)\, \mathrm{e}^{-i\overline{w}x}\, d\mu(w).$$

The operator R is unitary, and its inverse $R^*\colon L_2(\mathbb{R}_+) \to \mathcal{A}^2(\Pi)$ is given by

$$(R^* f)(z) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} \sqrt{\xi} f(\xi)\, \mathrm{e}^{iz\xi}\, d\xi.$$

The next theorem gives a criterion for an operator to be vertical, and is an analogue of the Zorboska result [15] for radial operators.

Theorem 2.3. *Let $S \in \mathcal{L}(\mathcal{A}^2(\Pi))$. The following conditions are equivalent:*

(a) S is invariant under horizontal shifts:

$$\forall h \in \mathbb{R} \qquad SH_h = H_h S.$$

(b) RSR^* is invariant under multiplication by Θ_h^+ for all $h \in \mathbb{R}$:

$$\forall h \in \mathbb{R} \qquad RSR^* M_{\Theta_h^+} = M_{\Theta_h^+} RSR^*.$$

(c) There exists a function $\sigma \in L_\infty(\mathbb{R}_+)$ such that

$$S = R^* M_\sigma R.$$

(d) The Berezin transform of S is a vertical function, i.e., depends on $\operatorname{Im}(w)$ *only.*

Proof. $(a) \Rightarrow (b)$. Follows from the formulas $R^* M_{\Theta_h^+} = H_h R^*$ and $RH_h = M_{\Theta_h^+} R$.

$(b) \Rightarrow (c)$. Follows from Lemma 2.2.

$(c) \Rightarrow (d)$. Using the residue theorem we get

$$(RK_{\Pi,w})(x) = -i \frac{\sqrt{x}}{\sqrt{\pi}}\, \mathrm{e}^{-i \operatorname{Re}(w) x}\, \mathrm{e}^{-\operatorname{Im}(w) x}.$$

Therefore

$$\mathcal{B}(S)(w) = \frac{\langle M_\sigma R K_{\Pi,w}, RK_{\Pi,w} \rangle}{\langle K_{\Pi,w}, K_{\Pi,w} \rangle} = (2 \operatorname{Im}(w))^2 \int_0^{+\infty} x \sigma(x)\, \mathrm{e}^{-2 \operatorname{Im}(w) x}\, dx,$$

and $\mathcal{B}(S)(w)$ depends only on $\operatorname{Im}(w)$.

$(d) \Rightarrow (a)$. Compute the Berezin transform of $H_{-h} S H_h$ using the formula $H_h K_{\Pi,w} = K_{\Pi,w+h}$:

$$\begin{aligned} \mathcal{B}(H_{-h} S H_h)(w) &= \frac{\langle S H_h K_{\Pi,w}, H_h K_{\Pi,w} \rangle}{\|K_{\Pi,w}\|^2} = \frac{\langle S K_{\Pi,w+h}, K_{\Pi,w+h} \rangle}{\|K_{\Pi,w+h}\|^2} \\ &= \mathcal{B}(S)(w+h) = \mathcal{B}(S)(w). \end{aligned}$$

Since the Berezin transform is injective [8], $H_{-h} S H_h = S$. □

Corollary 2.4. *The set of all vertical operators on $\mathcal{L}(\mathcal{A}^2(\Pi))$ is a commutative C^*-algebra which is isometrically isomorphic to $L_\infty(\mathbb{R}_+)$.*

3 Vertical Toeplitz operators

In this section we establish necessary and sufficient conditions for a Toeplitz operator to be vertical.

Lemma 3.1. *Let $g \in L_\infty(\Pi)$. Then T_g is zero if and only if $g = 0$ almost everywhere.*

Proof. The corresponding result for Toeplitz operators on the Bergman space on the unit disk is well known, see, for example, [14, Theorem 2.8.2]. To extend it to the upper half-plane case, we introduce the Cayley transform

$$\psi\colon \Pi \to \mathbb{D}, \qquad w \longmapsto \frac{w-i}{w+i},$$

the corresponding unitary operator

$$U\colon \mathcal{A}^2(\mathbb{D}) \to \mathcal{A}^2(\Pi), \qquad f \longmapsto (f \circ \psi)\psi',$$

and observe that $U^* T_g U = T_{g\circ\psi^{-1}}$. □

The next elementary lemma gives a criterion for a function on $\mathbb{R}$ to be almost everywhere constant. We use there the Lebesgue measure in $\mathbb{R}^n$ for various dimensions ($n = 1, 2, 3$), indicating the dimension as a subindex: μ_n.

Lemma 3.2. *Let $f\colon \mathbb{R} \to \mathbb{C}$ be a measurable function. Then the following conditions are equivalent:*

(a) There exists a constant $c \in \mathbb{C}$ such that $f(x) = c$ for almost all $x \in \mathbb{R}$.

(b) $\mu_2(D) = 0$, where $D := \big\{(x, y) \in \mathbb{R}^2 \mid f(x) \ne f(y)\big\}$.

(c) $\mu_1(D_x) = 0$ for almost all $x \in \mathbb{R}$, where $D_x := \big\{y \in \mathbb{R} \mid f(x) \ne f(y)\big\}$.

Proof. (a)⇒(b). Let $C = \{x \in \mathbb{R} \mid f(x) \ne c\}$. The condition (a) means that $\mu_1(C) = 0$. Since $D \subset (C \times \mathbb{R}) \cup (\mathbb{R} \times C)$, we obtain $\mu_2(D) = 0$.

(b)⇒(c). Apply Tonelli's theorem to the characteristic function of D.

(c)⇒(a). Choose a point $x_0 \in \mathbb{R}$ such that $\mu_1(D_{x_0}) = 0$ and set $c := f(x_0)$. Then $f = c$ almost everywhere. □

Old proof of (b)⇒(c). Denote by Φ the characteristic function of D. By Tonelli's theorem,

$$\int_{\mathbb{R}} \mu_1(D_x)\,dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Phi(x, y)\,dy \right) dx = \int_{\mathbb{R}^2} \Phi\,d\mu = \mu_2(D) = 0,$$

and $\mu_1(D_x) = 0$ for almost all $x \in \mathbb{R}$. □

Proposition 3.3. *Let $g \in L_\infty(\Pi)$. The operator T_g is vertical if and only if there exists a function $a \in L_\infty(\mathbb{R}_+)$ such that $g(w) = a(\operatorname{Im}(w))$ for almost every $w \in \Pi$.*

Proof. Sufficiency. For every $h \in \mathbb{R}$, define $g_h\colon \Pi \to \mathbb{C}$ by $g_h(w) = g(w+h)$. Then for almost all $w \in \mathbb{C}$

$$g_h(w) = g(w+h) = a(\operatorname{Im}(w+h)) = a(\operatorname{Im}(w)) = g(w).$$

Applying the formula $H_{-h}T_gH_h = T_{g_h}$ we see that T_g is invariant with respect to horizontal translations.

Necessity. Since T_g is vertical, for every $h \in \mathbb{R}$ we have $T_g = H_{-h}T_gH_h = T_{g_h}$. By Lemma 3.1, $g = g_h$ almost everywhere. It means that for all $h \in \mathbb{R}$ the equality $\mu_2(E_h) = 0$ holds where

$$E_h := \big\{(u,v) \in \mathbb{R}^2 \mid g(u+h+iv) \neq g(u+iv)\big\}.$$

Define $\Lambda\colon \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{C}$ by

$$\Lambda(u,x,v) := \begin{cases} 0, & g(x+iv) = g(u+iv); \\ 1, & g(x+iv) \neq g(u+iv). \end{cases}$$

Then for all $h \in \mathbb{R}$

$$\big\{(u,v) \in \Pi \mid \Lambda(u,u+h,v) \neq 0\big\} = E_h$$

and by Tonelli's theorem

$$\begin{aligned} \int_{\mathbb{R}^2\times\mathbb{R}_+} \Lambda(u,x,v)\,d\mu_3(u,x,v) &= \int_{\mathbb{R}^2\times\mathbb{R}_+} \Lambda(u,u+h,v)\,d\mu_3(u,h,v) \\ &= \int_{\mathbb{R}} \left(\int_{\Pi} \Lambda(u,u+h,v)\,d\mu_2(u,v) \right) dh = \int_{\mathbb{R}} \mu_2(E_h)\,dh = 0. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}_+} \left(\int_{\mathbb{R}^2} \Lambda(u,x,v)\,d\mu_2(u,x) \right) dv = \int_{\mathbb{R}^2\times\mathbb{R}_+} \Lambda(u,x,v)\,d\mu_3(u,x,v) = 0,$$

and for almost $v \in \mathbb{R}_+$

$$\mu_2(\{(u,x) \in \mathbb{R}^2 \mid g(x+iv) \neq g(u+iv)\}) = \int_{\mathbb{R}^2} \Lambda(u,x,v)\,d\mu(u,x) = 0.$$

For such v, by Lemma 3.2, there exists a constant $c(v)$ such that $g(u+iv) = c(v)$. Then for $a\colon \mathbb{R}_+ \to \mathbb{C}$ defined by

$$a(v) = \begin{cases} c(v), & \text{if } \mu_2(\{(u,x) \in \mathbb{R}^2 \mid g(x+iv) \neq g(u+iv)\}) = 0, \\ 0, & \text{otherwise,} \end{cases}$$

we have $g(w) = a(\operatorname{Im}(w))$ for almost all $w \in \Pi$. □

We say that a measurable function $g\colon \Pi \to \mathbb{C}$ is *vertical* if there exists a measurable function $a\colon \mathbb{R}_+ \to \mathbb{C}$ such that $g(w) = a(\mathrm{Im}(w))$ for almost all w in Π.

The next result was proved in [11, Theorem 3.1] (see also [14, Theorem 5.2.1]).

Theorem 3.4. *Let $g(w) = a(\mathrm{Im}(w)) \in L_\infty$ be a vertical symbol. Then the Toeplitz operator T_g acting on $\mathcal{A}^2(\Pi)$ is unitary equivalent to the multiplication operator $M_{\gamma_a} = RT_gR^*$ acting on $L_2(\mathbb{R}_+)$. The function $\gamma_a = \gamma_a(s)$ is given by*

$$\gamma_a(s) := 2s \int_0^\infty a(t)\, \mathrm{e}^{-2ts}\, dt, \qquad s \in \mathbb{R}_+. \tag{3.1}$$

In particular, this implies that the C^*-algebra generated by vertical Toeplitz operators with bounded symbols is commutative and is isometrically isomorphic to the C^*-algebra generated by the set

$$\Gamma := \big\{\gamma_a \mid a \in L_\infty(\mathbb{R}_+)\big\}.$$

4 Very slowly oscillating functions on $\mathbb{R}_+$

In this section we introduce and discuss the algebra $\mathrm{VSO}(\mathbb{R}_+)$ of very slowly oscillating functions, and show that for any vertical symbol $a \in L_\infty(\mathbb{R}_+)$, the associated "spectral function" γ_a belongs to $\mathrm{VSO}(\mathbb{R}_+)$.

We introduce the logarithmic metric on the positive half-line by

$$\rho(x,y) := \big|\ln(x) - \ln(y)\big| \;:\; \mathbb{R}_+ \times \mathbb{R}_+ \to [0,+\infty).$$

It is easy to see that ρ is indeed a metric and that ρ is *invariant under dilations*: for all $x, y, t \in \mathbb{R}_+$,

$$\rho(tx,ty) = \rho(x,y).$$

Recall that the *modulus of continuity* of a function $f\colon \mathbb{R}_+ \to \mathbb{C}$ with respect to the metric ρ is defined for all $\delta > 0$ as

$$\omega_{\rho,f}(\delta) := \sup\big\{|f(x) - f(y)| \mid \quad x,y \in \mathbb{R}_+, \quad \rho(x,y) \le \delta\big\}.$$

Definition 4.1. Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ be a bounded function. We say that f is *very slowly oscillating* if it is uniformly continuous with respect to the metric ρ or, equivalently, if the composition $f \circ \exp$ is uniformly continuous with respect the usual metric on $\mathbb{R}$. Denote by $\mathrm{VSO}(\mathbb{R}_+)$ the set of such functions.

Proposition 4.2. $\mathrm{VSO}(\mathbb{R}_+)$ *is a closed C^*-algebra of the C^*-algebra $C_b(\mathbb{R}_+)$ of bounded continuous functions $\mathbb{R}_+ \to \mathbb{C}$ with pointwise operations.*

Proof. Using the following elementary properties of the modulus of continuity one can see that $\mathrm{VSO}(\mathbb{R}_+)$ is closed with respect to the pointwise operations:

$$\omega_{\rho,f+g} \le \omega_{\rho,f} + \omega_{\rho,g}, \qquad \omega_{\rho,fg} \le \|f\|_\infty \omega_{\rho,g} + \|g\|_\infty \omega_{\rho,f},$$
$$\omega_{\rho,\lambda f} = |\lambda| \omega_{\rho,f}, \qquad \omega_{\rho,f^*} = \omega_{\rho,f}.$$

The inequality $\omega_{\rho,f}(\delta) \le 2\|f-g\|_\infty + \omega_{\rho,g}(\delta)$ and the usual "$\frac{\varepsilon}{3}$-argument" show that $\mathrm{VSO}(\mathbb{R}_+)$ is topologically closed in $C_b(\mathbb{R}_+)$. $\square$

Note that instead of the logarithmic metric ρ we can use an alternative one: Let $\rho_1 \colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty)$ be defined by

$$\rho_1(x,y) := \frac{|x-y|}{\max(x,y)}.$$

It is easy to see that ρ_1 is a metric. To prove the triangle inequality $\rho_1(x,z) + \rho_1(z,y) - \rho_1(x,y) \ge 0$, use the symmetry between x and y and consider three cases: $x < y < z$, $x < z < y$, $z < x < y$. For example, if $x < y < z$, then

$$\rho_1(x,z) + \rho_1(z,y) - \rho_1(x,y) = \frac{(z-y)(x+y)}{yz} > 0.$$

The other two cases are considered analogously.

Lemma 4.3. *For every $x, y \in \mathbb{R}_+$ the following inequality holds*

$$\rho_1(x,y) \le \rho(x,y). \tag{4.1}$$

Proof. The metrics ρ and ρ_1 can be written in terms of max and min as shown below:

$$\rho(x,y) = \ln \frac{\max(x,y)}{\min(x,y)}, \qquad \rho_1(x,y) = 1 - \frac{\min(x,y)}{\max(x,y)}.$$

Since $\ln(u) \ge 1 - \dfrac{1}{u}$ for all $u \ge 1$, the substitution $u = \dfrac{\max(x,y)}{\min(x,y)}$ yields (4.1). $\square$

It can be proved that $\rho(x,y) \le 2\ln(2)\rho_1(x,y)$ if $\rho_1(x,y) < 1/2$. Thus $\mathrm{VSO}(\mathbb{R}_+)$ could be defined alternatively as the class of all bounded functions that are uniformly continuous with respect to ρ_1.

Theorem 4.4. *Let $a \in L_\infty(\mathbb{R}_+)$. Then $\gamma_a \in \mathrm{VSO}(\mathbb{R}_+)$. More precisely,*

$$\|\gamma_a\|_\infty \le \|a\|_\infty,$$

and γ_a is Lipschitz continuous with respect to the distance ρ:

$$|\gamma_a(y) - \gamma_a(x)| \le 2\rho(x,y)\|a\|_\infty, \tag{4.2}$$

that is

$$\omega_{\gamma_a}(\delta) \le 2\delta\|a\|_\infty. \tag{4.3}$$

Proof. The upper bound $\|\gamma_a\|_\infty \le \|a\|_\infty$ follows directly from the definition (3.1) of γ_a. The proof of (4.3) written below is based on an idea communicated to us by K. M. Esmeral García. First, we bound $|a(v)|$ by $\|a\|_\infty$:

$$|\gamma_a(x) - \gamma_a(y)| \le \|a\|_\infty \int_0^\infty \left|2vx\,\mathrm{e}^{-2vx} - 2vy\,\mathrm{e}^{-2vy}\right| \frac{dv}{v}.$$

Without lost of generality assume $y > x$, so the inequality

$$2vx\,\mathrm{e}^{-2vx} - 2vy\,\mathrm{e}^{-2vy} \ge 0$$

is true if and only if $v \ge v_0 := \frac{1}{2}\frac{1}{y-x}\ln\frac{y}{x}$. Then

$$\begin{aligned}|\gamma_a(x) - \gamma_a(y)| &\le \|a\|_\infty \int_0^{v_0} (2vy\,\mathrm{e}^{-2vy} - 2vx\,\mathrm{e}^{-2vx})\frac{dv}{v} \\ &\qquad + \|a\|_\infty \int_{v_0}^\infty (2vx\,\mathrm{e}^{-2vx} - 2vy\,\mathrm{e}^{-2vy})\frac{dv}{v} \\ &= 2\|a\|_\infty\,\mathrm{e}^{-2v_0 x}\left(1 - \mathrm{e}^{2v_0(x-y)}\right) \\ &\le 2\|a\|_\infty \rho_1(x,y) \le 2\|a\|_\infty \rho(x,y),\end{aligned}$$

where the last inequality uses Lemma 4.3. □

5 Density of Γ in $\mathrm{VSO}(\mathbb{R}_+)$

The set $\mathbb{R}_+$ provided with the standard multiplication and topology is a commutative locally compact topological group, whose Haar measure is given by $d\nu(s) := \frac{ds}{s}$.

For each $n \in \mathbb{N} := \{1, 2, \ldots\}$, we define a function $\psi_n \colon \mathbb{R}_+ \to \mathbb{C}$ by

$$\psi_n(s) = \frac{1}{\mathrm{B}(n,n)} \frac{s^n}{(1+s)^{2n}},$$

where B is the Beta function.

Proposition 5.1. *The sequence $(\psi_n)_{n=1}^\infty$ is a Dirac sequence, i.e., it satisfies the following three conditions:*

(a) For each $n \in \mathbb{N}$ and every $s \in \mathbb{R}_+$,

$$\psi_n(s) \ge 0.$$

(b) For each $n \in \mathbb{N}$,

$$\int_0^\infty \psi_n(s) \frac{ds}{s} = 1.$$

(c) For every $\delta > 0$,

$$\lim_{n\to\infty} \int_{\rho(s,1)>\delta} \psi_n(s) \frac{ds}{s} = 0.$$

Proof. The property (a) is obvious, and (b) follows from the formula for the Beta function:

$$B(x,y) = \int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\, ds.$$

We prove (c). Fix a $\delta > 0$. The function $s \mapsto \dfrac{s^{n-1}}{(1+s)^{2n}}$ reaches its maximum at the point $s_n := \frac{n-1}{n+1}$. It increases on the interval $[0, s_n]$ and decreases on the interval $[s_n, \infty)$. Since $s_n \to 1$, there exists a number $N \in \mathbb{N}$ such that $\mathrm{e}^{-\delta} < s_N$. Let $n \in \mathbb{N}$ with $n \geq N$. Then $\mathrm{e}^{-\delta} \leq s_N \leq s_n$, and for all $s \in (0, \mathrm{e}^{-\delta}]$ we obtain

$$\frac{s^{n-1}}{(1+s)^{2n}} \leq \frac{(\mathrm{e}^{-\delta})^{n-1}}{(1+\mathrm{e}^{-\delta})^{2n}}.$$

Integration of both sides from 0 to $\mathrm{e}^{-\delta}$ yields

$$\int_0^{\mathrm{e}^{-\delta}} \frac{s^{n-1}}{(1+s)^{2n}}\, ds \leq \left(\frac{\mathrm{e}^{-\delta}}{(1+e^{-\delta})^2} \right)^n = \left(\frac{1}{4\cosh^2(\delta/2)} \right)^n.$$

Applying Stirling's formula ([3, formula 8.327]), we have

$$\frac{1}{\mathrm{B}(n,n)} = \frac{\Gamma(2n)}{(\Gamma(n))^2} \sim \frac{n}{2} \frac{4^n}{\sqrt{\pi n}}.$$

Since $\cosh(\delta/2) > 1$,

$$\int_0^{\mathrm{e}^{-\delta}} \psi_n(t) \frac{dt}{t} \leq \frac{1}{\mathrm{B}(n,n)} \left(\frac{1}{4\cosh^2(\delta/2)} \right)^n \sim \frac{\sqrt{n}}{2\sqrt{\pi}\cosh^{2n}(\delta/2)} \to 0.$$

To prove a similar result for the integral from e^δ to ∞, make the change of variable $s = 1/t$:

$$\lim_{n\to\infty} \int_{\mathrm{e}^\delta}^\infty \psi_n(t)\, \frac{dt}{t} = \lim_{n\to\infty} \int_0^{\mathrm{e}^{-\delta}} \psi_n(s)\, \frac{ds}{s}.$$

Let

$$R_{n,\delta} := \int_{\rho(s,1)>\delta} \psi_n(s)\, \frac{ds}{s}, \tag{5.1}$$

then

$$\lim_{n\to\infty} R_{n,\delta} = \lim_{n\to\infty} \int_0^{\mathrm{e}^{-\delta}} \psi_n(s)\, \frac{ds}{s} + \lim_{n\to\infty} \int_{\mathrm{e}^\delta}^\infty \psi_n(s) \frac{ds}{s} = 0. \qquad \square$$

Introduce now the standard *Mellin convolution* of two functions a and b from $L_1(\mathbb{R}_+, d\nu)$:

$$(a * b)(x) := \int_0^\infty a(y) b\left(\frac{x}{y}\right) \frac{dy}{y}, \qquad x \in \mathbb{R}_+, \tag{5.2}$$

being a commutative and associative binary operation on $L_1(\mathbb{R}_+, d\nu)$.

Note that (5.2) is well defined also if one of the functions a or b belongs to $L_\infty(\mathbb{R}_+)$ and the other belongs to $L_1(\mathbb{R}_+, d\nu)$. In that case $a * b \in L_\infty(\mathbb{R}_+)$ and $a * b = b * a$. The associativity law also holds for any three functions a, b, c such that one of them belongs to $L_\infty(\mathbb{R}_+)$ and the other two belong to $L_1(\mathbb{R}_+, d\nu)$.

The next result is a special case of a well–known general fact on Dirac sequences and uniformly continuous functions on locally compact groups. For the reader's convenience we write a proof for our case.

Theorem 5.2. *Let* $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$. *Then*

$$\lim_{n\to\infty} \|\sigma * \psi_n - \sigma\|_\infty = 0. \tag{5.3}$$

Proof. For every $n \in \mathbb{N}$, $\delta > 0$ and $x \in \mathbb{R}_+$,

$$\begin{aligned} |(\sigma * \psi_n)(x) - \sigma(x)| &= \left| \int_0^\infty \sigma\left(\frac{x}{y}\right) \psi_n(y) \frac{dy}{y} - \int_0^\infty \sigma(x) \psi_n(y) \frac{dy}{y} \right| \\ &\le \int_0^\infty \left| \sigma\left(\frac{x}{y}\right) - \sigma(x) \right| \psi_n(y) \frac{dy}{y} = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{\rho(y,1)\le\delta} \left| \sigma\left(\frac{x}{y}\right) - \sigma(x) \right| \psi_n(y) \frac{dy}{y}, \\ I_2 &= \int_{\rho(y,1)>\delta} \left| \sigma\left(\frac{x}{y}\right) - \sigma(x) \right| \psi_n(y) \frac{dy}{y}. \end{aligned}$$

If $\rho(y,1) \le \delta$, then $\rho(x/y, x) = \rho(x, xy) = \rho(y, 1) \le \delta$. Thus

$$I_1 \le \omega_{\rho,\sigma}(\delta) \int_{\mathbb{R}} \psi_n(y) \frac{dy}{y} = \omega_{\rho,\sigma}(\delta).$$

For the term I_2 we obtain an upper bound in terms of $R_{n,\delta}$, see (5.1):

$$I_2 \le 2\|\sigma\|_\infty \int_{\rho(y,1)>\delta} \psi_n(y) \frac{dy}{y} = 2\|\sigma\|_\infty R_{n,\delta}.$$

Therefore

$$\|\sigma * \psi_n - \sigma\|_\infty \le \omega_{\rho,\sigma}(\delta) + 2\|\sigma\|_\infty R_{n,\delta}.$$

Given $\varepsilon > 0$, first apply the hypothesis that $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$ and choose $\delta > 0$ such that $\omega_{\rho,\sigma}(\delta) < \frac{\varepsilon}{2}$. Then use the fact that $R_{n,\delta} \to 0$ and find a number $N \in \mathbb{N}$ such that $R_{n,\delta} < \frac{\varepsilon}{4\|\sigma\|_\infty}$ for all $n \geq N$. Then for all $n \geq N$

$$\|\sigma * \psi_n - \sigma\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Recall now that, for each $m, n \in \mathbb{N}$, the *generalized Laguerre polynomial* (called also *associated Laguerre polynomial*) is defined by

$$L_n^{(m)}(t) = \frac{1}{n!}\, t^{-m}\, \mathrm{e}^t\, \frac{d^n}{dt^n}\Big(\mathrm{e}^{-t}\, t^{n+m}\Big) = \sum_{j=0}^{n} \frac{(-1)^j (n+m)!}{(n-j)!\,(m+j)!\,j!}\, t^j, \quad t \in \mathbb{R}_+.$$

Then, for each $n \in \mathbb{N}$, we introduce the function $\phi_n \colon \mathbb{R}_+ \to \mathbb{C}$ by

$$\phi_n(t) = \frac{1}{(n-1)!}\, t^n\, \mathrm{e}^{-t}\, L_{n-1}^{(n)}(t). \tag{5.4}$$

Each function ϕ_n is obviously bounded and continuous on $\mathbb{R}_+$, and admits the following alternative representation

$$\phi_n(t) = \frac{1}{\big((n-1)!\big)^2}\, \frac{d^{n-1}}{dt^{n-1}}\left(\mathrm{e}^{-t}\, t^{2n-1}\right).$$

The next lemma relates the functions ψ_n and ϕ_n via the Laplace transform $\mathcal{L}$, which is defined by

$$\mathcal{L}(f)(s) := \int_0^\infty f(t)\,\mathrm{e}^{-st}\, dt.$$

Lemma 5.3. *For each $n \in \mathbb{N}$,*

$$\frac{\psi_n(s)}{s} = \mathcal{L}(\phi_n)(s), \qquad s \in \mathbb{R}_+. \tag{5.5}$$

Proof. The function $t \mapsto \mathrm{e}^{-t}\, t^{2n-1}$ and its first $2n-2$ derivatives vanish at 0 and $+\infty$. Integrating by parts $n-1$ times we get

$$\int_0^\infty \frac{d^{n-1}}{dt^{n-1}}\left(\mathrm{e}^{-t}\, t^{2n-1}\right)\mathrm{e}^{-st}\, dt = s^{n-1}\int_0^\infty \mathrm{e}^{-t}\, t^{2n-1}\, \mathrm{e}^{-st}\, dt = \frac{s^{n-1}\Gamma(2n)}{(1+s)^{2n}}.$$

Therefore

$$\mathcal{L}(\phi_n)(s) = \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}\, \frac{s^{n-1}}{(1+s)^{2n}} = \frac{\psi_n(s)}{s}. \qquad \square$$

Given a function $a\colon \mathbb{R}_+ \to \mathbb{C}$, we define $\widetilde{a}\colon \mathbb{R}_+ \to \mathbb{C}$ as $\widetilde{a}(t) = a(1/t)$.
The mapping $a \mapsto \widetilde{a}$ is obviously an involution:

$$\widetilde{\widetilde{a}} = a, \tag{5.6}$$

and, for all $a \in L_\infty(\mathbb{R}_+)$ and $b \in L_1(\mathbb{R}_+, d\nu)$, we have

$$\widetilde{a * b} = \widetilde{a} * \widetilde{b}. \tag{5.7}$$

The change of variable $t = \frac{1}{u}$ yields

$$\int_0^\infty a(t)b(st)\,\frac{dt}{t} = (\widetilde{a} * b)(s). \tag{5.8}$$

The next lemma relates "spectral functions" γ_a with Mellin convolutions.

Lemma 5.4. *Let $\alpha(u) = 2u\,\mathrm{e}^{-2u}$, then for each $a \in L_\infty(\mathbb{R}_+)$,*

$$\gamma_a = \widetilde{a} * \alpha. \tag{5.9}$$

Proof. Rewrite γ_a in the form

$$\gamma_a(s) = \int_0^\infty a(t)\,\left(2st\,\mathrm{e}^{-2st}\right)\frac{dt}{t}$$

and apply (5.8). □

Introduce the function $m_2(s) := 2s$, then (5.5) and (5.9) imply that the elements ψ_n of the Dirac sequence are in fact certain "spectral functions":

$$\psi_n = (\widetilde{\phi_n \circ m_2}) * \alpha = \gamma_{\phi_n \circ m_2}.$$

Now we are ready to prove the main result of the paper.
Recall first that, by Theorem 3.4, the C^*-algebra generated by vertical Toeplitz operators with bounded symbols is isometrically isomorphic to the C^*-algebra generated by the set

$$\Gamma = \left\{\gamma_a \mid a \in L_\infty(\mathbb{R}_+)\right\}.$$

Theorem 5.5. *We have that* $\overline{\Gamma} = \mathrm{VSO}(\mathbb{R}_+)$.

Proof. Let $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$. For each $n \in \mathbb{N}$, we define $a_n\colon \mathbb{R}_+ \to \mathbb{C}$ by

$$a_n := \widetilde{\sigma} * (\phi_n \circ m_2).$$

From (5.4) it follows that $\phi_n \in L_1(\mathbb{R}_+, d\nu)$, and thus $a_n \in L_\infty(\mathbb{R}_+)$. Then equations (5.7), (5.6) and the associativity of Mellin convolutions yield

$$\gamma_{a_n} = \widetilde{a_n} * \alpha = \left(\widetilde{\widetilde{\sigma}} * (\widetilde{\phi_n \circ m_2})\right) * \alpha = \sigma * \left((\widetilde{\phi_n \circ m_2}) * \alpha\right) = \sigma * \psi_n,$$

which means that $\sigma_n * \psi_n \in \Gamma$. To finish the proof apply Theorem 5.2. □

Let us mention some important corollaries of the theorem. First of all it implies that the C^*-algebra $\mathcal{VT}(L_\infty)$ generated by Toeplitz operators with bounded vertical symbols is isometrically isomorphic to $\mathrm{VSO}(\mathbb{R}_+)$. Moreover it shows that the set of initial generators of $\mathcal{VT}(L_\infty)$ (i.e., the Toeplitz operators with bounded vertical symbols) is dense in $\mathcal{VT}(L_\infty)$. That is, the two quite different types of the closures, the C^*-algebraic closure and the topological closure, of the set of initial generators end up with the same result: the C^*-algebra $\mathcal{VT}(L_\infty)$ generated by Toeplitz operators with bounded vertical symbols.

Then, the theorem permits us to compare and realize the difference between the algebra generated by general vertical operators and its subalgebra generated by special vertical operators, Toeplitz operators with bounded vertical symbols. The first one is isomorphic to $L_\infty(\mathbb{R}_+)$, while the second, its subalgebra, is isomorphic to $\mathrm{VSO}(\mathbb{R}_+)$.

In this connection it is interesting to consider "intermediate", in a sense, operators, the *bounded* vertical Toeplitz operators whose defining symbols are *unbounded*. As it turns out such operators *do not* necessarily belong to the algebra $\mathcal{VT}(L_\infty)$ generated by vertical Toeplitz operators with *bounded* symbols.

The next section is devoted to an example of such an operator.

6 Example

Note that γ_a can be defined by the formula (3.1) not only if $a \in L_\infty(\mathbb{R}_+)$, but also if $a \in L_1(\mathbb{R}_+, \mathrm{e}^{-\eta t}\,dt)$ for all $\eta > 0$.

In this section we construct a non-bounded function $a\colon \mathbb{R}_+ \to \mathbb{C}$ such that $a \in L_1(\mathbb{R}_+, \mathrm{e}^{-\eta t}\,dt)$ for all $\eta > 0$ and $\gamma_a \in L_\infty(\mathbb{R}_+)$, but $\gamma_a \notin \mathrm{VSO}(\mathbb{R}_+)$. This implies that the corresponding vertical Toeplitz operator is bounded, but it does not belong to the C^*-algebra generated by vertical Toeplitz operators with bounded generating symbols.

The idea of this example is taken from [4].

Proposition 6.1. *Define* $f\colon \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\} \to \mathbb{C}$ *by*

$$f(z) := \frac{1}{z+1} \exp\left(\frac{i}{3\pi} \ln^2(z+1)\right), \tag{6.1}$$

where $\ln$ *is the principal value of the natural logarithm (with imaginary part in* $(-\pi, \pi]$*). Then there exists a unique function* $A\colon \mathbb{R}_+ \to \mathbb{C}$ *such that* $A \in L_1(\mathbb{R}_+, \mathrm{e}^{-\eta u}\,du)$ *for all* $\eta > 0$ *and* f *is the Laplace transform of* A*:*

$$f(z) = \int_0^{+\infty} A(u)\,\mathrm{e}^{-zu}\,du.$$

Proof. For every $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$ we write $\ln(z+1)$ as $\ln|z+1| + i\arg(z+1)$ with

$-\frac{\pi}{2} < \arg(z+1) < \frac{\pi}{2}$. Then

$$\begin{aligned}|f(z)| &= \frac{1}{|z+1|}\left|\exp\left(\frac{i}{3\pi}\big(\ln|z+1| + i\arg(z+1)\big)^2\right)\right| \\ &= \frac{1}{|z+1|}\exp\left(-\frac{2\arg(z+1)}{3\pi}\ln|z+1|\right) \\ &= \frac{1}{|z+1|^{1+\frac{2\arg(z+1)}{3\pi}}}.\end{aligned}$$

Since $|z+1| \geq 1$ and $-\frac{1}{3} < -\frac{2\arg(z+1)}{3\pi} < \frac{1}{3}$,

$$|f(z)| \leq \frac{1}{|z+1|^{2/3}}.$$

Therefore for every $x > 0$,

$$\int_{\mathbb{R}} |f(x+iy)|^2\,dy \leq \int_{\mathbb{R}} \frac{dy}{((x+1)^2+y^2)^{2/3}} < \int_{\mathbb{R}} \frac{dy}{(1+y^2)^{2/3}} < +\infty,$$

and f belongs to the Hardy class H^2 on the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. By Paley–Wiener theorem (see, for example, Rudin [6, Theorem 19.2]), there exists a function $A \in L_2(\mathbb{R}_+)$ such that for all $x > 0$

$$f(x) = \int_0^{+\infty} A(u)\,\mathrm{e}^{-ux}\,du.$$

The uniqueness of A follows from the injective property of the Laplace transform. Applying Hölder's inequality we easily see that $A \in L_1(\mathbb{R}_+, \mathrm{e}^{-\eta u}\,du)$ for all $\eta > 0$:

$$\int_0^{+\infty} |A(u)|\,\mathrm{e}^{-\eta u}\,du \leq \|A\|_2 \left(\int_0^{+\infty} \mathrm{e}^{-2\eta u}\,du\right)^{1/2} = \frac{\|A\|_2}{\sqrt{2\eta}}.$$

□

Proposition 6.2. *The function $\sigma\colon \mathbb{R}_+ \to \mathbb{C}$ defined by*

$$\sigma(s) := \frac{s}{s+1}\exp\left(\frac{i}{3\pi}\ln^2(s+1)\right), \tag{6.2}$$

belongs to $L_\infty(\mathbb{R}_+) \setminus \mathrm{VSO}(\mathbb{R}_+)$. Moreover there exists a function $a\colon \mathbb{R}_+ \to \mathbb{C}$ such that $a \in L_1(\mathbb{R}_+, \mathrm{e}^{-\eta t}\,dt)$ for all $\eta > 0$ and $\sigma = \gamma_a$.

Proof. The function σ is bounded since $|\sigma(s)| \leq \frac{s}{s+1} \leq 1$ for all $s \in \mathbb{R}_+$. Let A be the function from Proposition 6.1. Define $a\colon \mathbb{R}_+ \to \mathbb{C}$ by

$$a(s) = A(2s).$$

Then for all $\eta > 0$

$$\int_0^{+\infty} |a(t)| \,\mathrm{e}^{-\eta t}\, du = \frac{1}{2} \int_0^{+\infty} |A(t)| \,\mathrm{e}^{-\eta t/2}\, dt < +\infty,$$

and

$$\begin{aligned}\gamma_a(s) &= 2s \int_0^{+\infty} a(t)\,\mathrm{e}^{-2st}\, dt = 2s \int_0^{+\infty} A(2t)\,\mathrm{e}^{-2st}\, dt \\ &= s \int_0^{+\infty} A(t)\,\mathrm{e}^{-st}\, dt = \frac{s}{s+1} \exp\left(\frac{i}{3\pi} \ln^2(s+1)\right) = \sigma(s).\end{aligned}$$

Let us prove that $\sigma \notin \mathrm{VSO}(\mathbb{R}_+)$. For all $s, t \in \mathbb{R}_+$

$$\begin{aligned}|\sigma(s) - \sigma(t)| &= \Bigg|\left(1 - \frac{1}{s+1}\right) \exp\left(\frac{i}{3\pi} \ln^2(s+1)\right) \\ &\qquad - \left(1 - \frac{1}{t+1}\right) \exp\left(\frac{i}{3\pi} \ln^2(t+1)\right)\Bigg| \\ &\geq \left|\exp\left(\frac{i}{3\pi} \ln^2(s+1)\right) - \exp\left(\frac{i}{3\pi} \ln^2(t+1)\right)\right| \\ &\qquad - \frac{1}{s+1} - \frac{1}{t+1} \\ &= \left|\exp\left(\frac{i}{3\pi}\big(\ln^2(s+1) - \ln^2(t+1)\big)\right) - 1\right| - \frac{1}{s+1} - \frac{1}{t+1}.\end{aligned}$$

Replace s by the following function of t:

$$s(t) := t + \frac{t+1}{\ln^{1/2}(t+1)}.$$

Then

$$\begin{aligned}\ln(s(t)+1) &= \ln(t+1) + \ln\left(1 + \frac{1}{\ln^{1/2}(t+1)}\right) \\ &= \ln(t+1) + \frac{1}{\ln^{1/2}(t+1)} - \frac{1}{2\ln(t+1)} + \mathcal{O}\left(\frac{1}{\ln^{3/2}(t+1)}\right).\end{aligned}$$

Denote $\ln^2(s(t)+1) - \ln^2(t+1)$ by L_t and consider the asymptotic behavior of L_t as $t \to +\infty$:

$$L_t := \ln^2(s(t)+1) - \ln^2(t+1) = -1 + 2\ln^{1/2}(t+1) + \mathcal{O}\left(\frac{1}{\ln(t+1)}\right).$$

Since L_t is continuous and tends to $+\infty$ as $t \to +\infty$, for every $T > 40$ there exists an integer $t \geq T$ such that $L_t + 1$ is equal to an integer multiple of $6\pi^2$, say to $6m\pi^2$:

$$L_t + 1 = 6m\pi^2.$$

For such t,

$$\left|\exp\left(\frac{i}{3\pi}L_t\right)-1\right| = \left|\exp\left(\frac{i}{3\pi}(6m\pi^2-1)\right)-1\right|$$
$$= \left|\exp\left(-\frac{i}{3\pi}\right)-1\right| \approx 0.106 > \frac{1}{10}$$

and

$$|\sigma(s(t))-\sigma(t)| \geq \left|\exp\left(\frac{i}{3\pi}L_t\right)-1\right| - \frac{2}{T+1} > \frac{1}{10} - \frac{1}{20} = \frac{1}{20}.$$

It means that $|\sigma(s(t))-\sigma(t)|$ does not converge to 0 as t goes to infinity. On the other hand,

$$\rho(s(t),t) = \ln\frac{s(t)}{t} \leq \frac{t+1}{t\ln^{1/2}(t+1)} \to 0.$$

Thus $\sigma \notin \mathrm{VSO}(\mathbb{R}_+)$. □

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Crispin Herrera Yañez
Departamento de Matemáticas, CINVESTAV
Apartado Postal 14-740, 07000, México, D.F., México
cherrera@math.cinvestav.mx

Egor A. Maximenko
Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional,
07730, México, D.F., México
maximenko@esfm.ipn.mx

Nikolai Vasilevski
Departamento de Matemáticas, CINVESTAV
Apartado Postal 14-740, 07000, México, D.F., México
nvasilev@math.cinvestav.mx