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Integral Equations and Operator Theory

Radial Toeplitz operators revisited: Discretization of the vertical case

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Abstract. It is known that radial Toeplitz operators acting on a weighted Bergman space of the analytic functions on the unit ball generate a commutative C*-algebra. This algebra has been explicitly described via its identification with the C*-algebra VSO($\mathbb{N}$) of bounded very slowly oscillating sequences (these sequences were used by R. Schmidt and other authors in Tauberian theory). On the other hand, it was recently proved that the C*-algebra generated by Toeplitz operators with bounded measurable vertical symbols is unitarily isomorphic to the C*-algebra VSO($\mathbb{R}_+$) of "very slowly oscillating functions", i.e. the bounded functions that are uniformly continuous with respect to the logarithmic distance $\rho(x, y) = |\ln(x) - \ln(y)|$. In this note we show that the results for the radial case can be easily deduced from the results for the vertical one.

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1. Introduction

Two classes of Toeplitz operators, *radial* and *vertical*, are discussed in the paper. Radial Toeplitz operators T_a, acting on the weighted Bergman space on the unit disk, or unit ball, are those whose bounded measurable symbols depend only on the radial part of the argument, i.e., $a(z) = a(|z|)$. Vertical Toeplitz operators T_b, acting on the weighted Bergman space on the upper

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half-plane, are those whose bounded measurable symbols depend only on the imaginary part of the argument, i.e., $b(z) = b(y)$, $z = x + iy$.

For the case of the unit disk (upper half-plane) they constitute two of the three model classes that generate commutative C^*-algebras of Toeplitz operators (see [17] for further details). In all cases the corresponding Toeplitz operators can be diagonalized. To be more precise we introduce necessary definitions.

For each $\lambda \in (-1, \infty)$ denote by $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ the weighted Bergman space consisting of the functions that are analytic on the unit ball $\mathbb{B}^n$ and square integrable with respect to the measure $dv_{n,\lambda}(z) = c_{n,\lambda}(1-|z|^2)^\lambda\, dv$, where dv is the standard Lebesgue measure on $\mathbb{C}^n$ and $c_{n,\lambda}$ is the normalizing constant such that $dv_{n,\lambda}(\mathbb{B}^n) = 1$. The *canonical basis* of the space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ consists of the normalized monomials

$$e_\alpha = c_\alpha z^\alpha, \qquad \alpha \in \mathbb{Z}^n_+,$$

where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.

Let $T_{a,n,\lambda}$ be the Toeplitz operator with radial generating symbol a acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. It is well known (see [10] for the one-dimensional case and [6] for the general case) that $T_{a,n,\lambda}$ is diagonal with respect to the basis $(e_\alpha)_{\alpha\in\mathbb{Z}^n_+}$:

$$T_{a,n,\lambda} e_\alpha = \beta_{a,n,\lambda}(|\alpha| + 1) e_\alpha, \qquad \alpha \in \mathbb{Z}^n_+,$$

where the corresponding eigenvalues depend only on the norm of the multi-indices and are of the form

$$\beta_{a,n,\lambda}(k) = \frac{1}{\mathrm{B}(n + k - 1, \lambda + 1)} \int_0^1 a(\sqrt{r}) r^{k+n-2}(1 - r)^\lambda\, dr, \qquad k \in \mathbb{N}. \tag{1.1}$$

Let $\mathrm{B}_{n,\lambda}$ stand for the set of all eigenvalue sequences $\beta_{a,n,\lambda}$ for generating symbols $a \in L_\infty([0, 1))$:

$$\mathrm{B}_{n,\lambda} := \big\{\beta_{a,n,\lambda} \colon\ a \in L_\infty([0, 1))\big\},$$

and let $\mathcal{A}_{n,\lambda}$ denote the C^*-subalgebra of $l_\infty(\mathbb{N})$ generated by $\mathrm{B}_{n,\lambda}$. The C^*-algebra generated by the operators $T_{a,n,\lambda}$, with $a \in L_\infty([0, 1))$, is obviously isometrically isomorphic to $\mathcal{A}_{n,\lambda}$.

As for the vertical Toeplitz operators on the upper half-plane, it was proved by Vasilevski [16], for the non-weighted case, and by Grudsky, Karapetyants, Vasilevski [4], for the weighted case, that such operators can be diagonalized via an appropriate unitary operator. The corresponding "spectral function" (an analogue of the sequence of eigenvalues) is given by

$$\gamma_{b,\lambda}(x) = \frac{x^{\lambda+1}}{\Gamma(\lambda + 1)} \int_0^{+\infty} b(v/2)\, \mathrm{e}^{-xv}\, v^\lambda\, dv, \qquad x \in \mathbb{R}_+. \tag{1.2}$$

At this stage an important question emerges: *Find an intrinsic* (*independent on Toeplitz operators*) *description of the C^*-algebra $\mathcal{A}_{n,\lambda}$ generated by spectral sequences* (1.1) *and of the C^*-algebra generated by the spectral functions* (1.2).

Surprisingly the answer returns us to the notions of the classical analysis of the beginning of the 20th century, in particular, to the notions of slowly oscillating sequences and functions.

The class of sequences satisfying $\sigma_j - \sigma_k \to 0$ as $j/k \to 1$ plays an important role in Tauberian theory and appeared in the literature since the article by Schmidt [13, § 9, Definition 10]; a similar condition for functions $[1, \infty) \to \mathbb{R}_+$ can be found in Landau [11, Satz X]. Note that Schmidt used the term *slowly oscillating sequences* (*langsam oszillierenden Folgen* in German). We prefer to call them *very slowly oscillating* sequences or functions as the term *slowly oscillating* is frequently used in other senses. We will denote such classes by $\mathrm{VSO}(\mathbb{N})$ and $\mathrm{VSO}(\mathbb{R}_+)$, respectively.

The complete answer for the radial case took about ten years and was based on a number of rather deep results. The principal step was made by Suárez. He constructed a powerful tool, the so-called n-Berezin transform [14], to approximate operators (satisfying some conditions) by Toeplitz operators. Using this technique he proved [15] that $\mathcal{A}_{1,0}$ coincides with the closure in $\ell_\infty(\mathbb{N})$ of the class $d_1(\mathbb{N})$ consisting of all bounded sequences σ such that

$$\sup_{n \in \mathbb{N}} n|\sigma_{n+1} - \sigma_n| < +\infty.$$

Grudsky, Maximenko and Vasilevski [7] identified the closure of $d_1(\mathbb{N})$ with the class of very slowly oscillating sequences $\mathrm{VSO}(\mathbb{N})$ and deduced from [15] that the set $\mathrm{B}_{n,0}$ is dense in $\mathrm{VSO}(\mathbb{N})$; therefore $\mathcal{A}_{n,0} = \mathrm{VSO}(\mathbb{N})$, i.e., the C*-algebra generated by radial Toeplitz operators on the unit ball $\mathbb{B}^n$ is isometrically isomorphic to $\mathrm{VSO}(\mathbb{N})$.

Inspired by Nam, Zheng, and Zhong [12], Bauer, Herrera Yañez, and Vasilevski [1, 2] introduced the (m, λ)-Berezin transform and generalized the above result to the weighted case. In particular, they proved that the set $\mathrm{B}_{n,\lambda}$ is dense in $\mathrm{VSO}(\mathbb{N})$, and therefore $\mathcal{A}_{n,\lambda} = \mathrm{VSO}(\mathbb{N})$, obtaining thus a characterization of $\mathcal{A}_{n,\lambda}$ for every $n \in \mathbb{N}$ and $\lambda > -1$.

The results for the vertical case are more recent. As was proved (in [8] for the non-weighted case and in [9] for the weighted case), the set $\{\gamma_{b,\lambda} \colon\ b \in L_\infty(\mathbb{R}_+)\}$ is dense in $\mathrm{VSO}(\mathbb{R}_+)$. The main idea was to express the integral transform (1.2) through the Mellin convolution and then apply an approximate identity.

The aim of this paper is to show that the results for the radial case can be easily deduced from the results for the vertical case, not involving heavy artillery of [1, 2, 7, 14, 15].

The paper is organized as follows. Section 2 contains auxiliary material. Here we define the classes $\mathrm{VSO}(\mathbb{R}_+)$ and $\mathrm{VSO}(\mathbb{N})$ of very slowly oscillating functions and sequences as the classes of functions and sequences that are uniformly continuous with respect to the logarithmic metric, and describe the interrelations between these classes. The core of the paper is Section 3. We recall here briefly the results for the vertical Toeplitz operators, and deduce from them the corresponding results for the radial Toeplitz operators. The main result is Theorem 3.6 which states that the set $\mathrm{B}_{n,\lambda}$ is dense in $\mathrm{VSO}(\mathbb{N})$.

2. Very slowly oscillating functions and sequences

We denote by $\mathbb{R}_+$ the positive half-line $\{x \in \mathbb{R}\colon\ x > 0\}$ and denote by ρ the logarithmic metric on $\mathbb{R}_+$:

$$\rho(x,y) := |\ln(x) - \ln(y)| = \left|\ln\left(\frac{x}{y}\right)\right| = \ln\frac{\max(x,y)}{\min(x,y)}, \qquad x,y > 0.$$

We will use as well the restriction $\rho_{\mathbb{N}}$ of the metric ρ to $\mathbb{N}$:

$$\rho_{\mathbb{N}}(j,k) := \rho(j,k), \qquad j,k \in \mathbb{N}.$$

Given a sequence $\sigma\colon \mathbb{N} \to \mathbb{C}$, let $\omega_{\rho,\sigma}$ be its *modulus of continuity* with respect to $\rho_{\mathbb{N}}$:

$$\omega_{\rho,\sigma}(\delta) := \sup\{|\sigma(j) - \sigma(k)|\colon\ j,k \in \mathbb{N},\ \rho(j,k) \le \delta\}.$$

Denote by $\mathrm{VSO}(\mathbb{N})$ the set of all bounded sequences $\sigma\colon \mathbb{N} \to \mathbb{C}$ that are uniformly continuous with respect to $\rho_{\mathbb{N}}$, i.e.,

$$\lim_{\delta\to 0} \omega_{\rho,\sigma}(\delta) = 0.$$

The set $\mathrm{VSO}(\mathbb{N})$ is obviously a C^*-subalgebra of $\ell_\infty(\mathbb{N})$.

Analogously, given a function $f\colon \mathbb{R}_+ \to \mathbb{C}$, we denote by $\Omega_{\rho,f}$ its *modulus of continuity* with respect to the metric ρ:

$$\Omega_{\rho,f}(\delta) := \sup\{|f(x) - f(y)|\colon\ x,y \in \mathbb{R}_+,\ \rho(x,y) \le \delta\};$$

and denote by $\mathrm{VSO}(\mathbb{R}_+)$ the set of all bounded functions $f\colon \mathbb{R}_+ \to \mathbb{C}$ that are uniformly continuous with respect to ρ, i.e.,

$$\lim_{\delta\to 0} \Omega_{\rho,f}(\delta) = 0.$$

It is easy to see [8] that $\mathrm{VSO}(\mathbb{R}_+)$ is a C^*-subalgebra of $C_b(\mathbb{R}_+)$, the algebra of bounded continuous functions on $\mathbb{R}_+$, equipped with the sup-norm.

Introduce as well

$$\rho_1(x,y) := \frac{|x-y|}{\max(x,y)} = 1 - \frac{\min(x,y)}{\max(x,y)}, \qquad x,y \in \mathbb{R}_+.$$

Proposition 2.1. *The expression ρ_1 is a metric on $\mathbb{R}_+$, which is uniformly equivalent to ρ. More precisely, for every $x,y \in \mathbb{R}_+$*

$$\rho_1(x,y) \le \rho(x,y) \tag{2.1}$$

and for every $x,y \in \mathbb{R}_+$, if $\rho_1(x,y) \le 1/2$, then

$$\rho(x,y) \le 2\ln(2)\rho_1(x,y). \tag{2.2}$$

Proof. Direct calculations, considering various arrangements of three points (see [7, Propositions 5]), prove the triangular inequality.

Setting $u = \max(x,y)/\min(x,y)$ in the equality $1 - \frac{1}{u} \le \ln(u)$ justifies (2.1). On the other hand, the function $t \mapsto -\frac{\ln(1-t)}{t}$ increases on $[0,1/2]$, therefore for every $t \in [0,1/2]$

$$\ln\frac{1}{1-t} \le 2\ln(2)t.$$

Substituting t by $\rho_1(x,y)$ we obtain (2.2). $\square$

To describe the relations between VSO($\mathbb{N}$) and VSO($\mathbb{R}_+$), we introduce the piecewise-linear extensions of sequences as follows.

Let $\sigma\colon \mathbb{N} \to \mathbb{C}$. Consider the function $f : \mathbb{R}_+ \to \mathbb{C}$ obtained from σ by the piecewise-linear interpolation:

$$f(x) := \begin{cases} \sigma_1, & x \in (0,1); \\ (j+1-x)\sigma_j + (x-j)\sigma_{j+1}, & x \in [j, j+1),\ j \in \mathbb{N}. \end{cases} \tag{2.3}$$

In what follows $\lfloor x \rfloor$ stands for the integer part of $x \in \mathbb{R}_+$.

Lemma 2.2. *Given $\sigma\colon \mathbb{N} \to \mathbb{C}$, we define f by (2.3). Then $\|f\|_\infty = \|\sigma\|_\infty$,*

$$|f(x) - f(y)| \le (y-x)\omega_{\rho,\sigma}(1), \qquad 0 < x < y \tag{2.4}$$

and

$$|f(x) - f(y)| \le 2\omega_{\rho,\sigma}(\rho(\lfloor x \rfloor, \lfloor y \rfloor + 1)), \qquad 1 \le x < y. \tag{2.5}$$

Proof. Put $\sigma_0 = \sigma_1$. Then (2.3) can be rewritten as

$$f(x) = (\lfloor x \rfloor + 1 - x)\sigma_{\lfloor x \rfloor} + (x - \lfloor x \rfloor)\sigma_{\lfloor x \rfloor + 1}.$$

Since the value of f at every point $x > 0$ is a convex combination of two values of the original sequence σ, the inequality $\|f\|_\infty \le \|\sigma\|_\infty$ holds. On the other hand, f is an extension of σ, therefore the inverse inequality is also true.

An elementary computation shows that if $s, t > 0$ and s, t belong to the same interval $[j, j+1]$ for some $j \in \{0, 1, 2, \ldots\}$, then

$$|f(s) - f(t)| = |t - s|\,|\sigma_j - \sigma_{j+1}|.$$

Since $|\sigma_j - \sigma_{j+1}| \le \omega_{\rho,\sigma}(\rho(j, j+1)) \le \omega_{\rho,\sigma}(1)$ for every $j \in \mathbb{N}$ and $\sigma_0 = \sigma_1$,

$$|f(s) - f(t)| \le |t - s|\,\omega_{\rho,\sigma}(1), \qquad \lfloor t \rfloor = \lfloor s \rfloor = j \in \mathbb{Z}_+. \tag{2.6}$$

To prove (2.4), assume that $0 < x < y$. The case $\lfloor x \rfloor = \lfloor y \rfloor$ is already covered by (2.6). If $\lfloor x \rfloor < \lfloor y \rfloor$, then insert intermediate integer points between x and y and apply (2.6) in each segment of this division:

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - \sigma_{\lfloor x \rfloor + 1}| + \sum_{j=\lfloor x \rfloor + 1}^{\lfloor y \rfloor - 1} |\sigma_j - \sigma_{j+1}| + |\sigma_{\lfloor y \rfloor} - f(y)| \\ &\le (\lfloor x \rfloor + 1 - x)\omega_{\rho,\sigma}(1) + (\lfloor y \rfloor - \lfloor x \rfloor - 1)\omega_{\rho,\sigma}(1) \\ &\quad + (y - \lfloor y \rfloor)\omega_{\rho,\sigma}(1) \\ &= (y - x)\omega_{\rho,\sigma}(1). \end{aligned}$$

To prove (2.5), suppose that $1 \le x < y$. Then

$$\begin{aligned} |f(x) - f(y)| &= |(1-u)\sigma_j + u\sigma_{j+1} - (1-v)\sigma_k - v\sigma_{k+1}| \\ &\le (1-u)|\sigma_j - \sigma_k| + u|\sigma_{j+1} - \sigma_{k+1}| + |u - v|\,|\sigma_k - \sigma_{k+1}| \\ &\le 2\omega_{\rho,\sigma}(\rho(j, k+1)). \end{aligned}$$

□

For every function $f \in \mathrm{VSO}(\mathbb{R}_+)$ we denote by $R(f)$ its restriction onto $\mathbb{N}$, and for every sequence $\sigma \in \mathrm{VSO}(\mathbb{N})$ we denote by $E(\sigma)$ its piecewise-linear extension defined in (2.3). Note that $R(E(\sigma)) = \sigma$ for every $\sigma \in \mathrm{VSO}(\mathbb{N})$.

Theorem 2.3. *The mapping* $R\colon \mathrm{VSO}(\mathbb{R}_+) \to \mathrm{VSO}(\mathbb{N})$ *is an epimorphism of* C^**-algebras. In particular, the set* $\mathrm{VSO}(\mathbb{N})$ *of sequences coincides with the set of the restrictions of functions from* $\mathrm{VSO}(\mathbb{R}_+)$:

$$\mathrm{VSO}(\mathbb{N}) = \{R(f)\colon\ f \in \mathrm{VSO}(\mathbb{R}_+)\}.$$

Proof. It is easy to see that $R(\mathrm{VSO}(\mathbb{R}_+)) \subseteq \mathrm{VSO}(\mathbb{N})$ and that R is a homomorphism. In order to prove that R is surjective, we start with $\sigma \in \mathrm{VSO}(\mathbb{N})$ and construct $f = E(\sigma)$, then $\|f\|_\infty = \|\sigma\|_\infty$. Considering two cases: $y - x < \sqrt{\delta}$ and $y - x \geq \sqrt{\delta}$, we prove first that for every $\delta \in (0, 1/4)$

$$\Omega_{\rho,f}(\delta) \leq \max\bigl(\sqrt{\delta}\,\omega_{\rho,\sigma}(1),\ 2\omega_{\rho,\sigma}(6\sqrt{\delta})\bigr). \tag{2.7}$$

Let $y - x < \sqrt{\delta}$, then by (2.4)

$$|f(x) - f(y)| \leq \sqrt{\delta}\,\omega_{\rho,\sigma}(1). \tag{2.8}$$

If $y - x \geq \sqrt{\delta}$, then

$$\delta \geq \rho(x, y) = \ln\frac{y}{x} \geq \frac{y - x}{y} \quad \text{and} \quad y \geq \frac{y - x}{\delta} \geq \frac{1}{\sqrt{\delta}}.$$

Moreover

$$x \geq y - y\delta \geq \frac{3y}{4} \geq \frac{3}{4\sqrt{\delta}}.$$

Therefore

$$x - 1 \geq \frac{3}{4\sqrt{\delta}} - 1 = \frac{3 - 4\sqrt{\delta}}{4\sqrt{\delta}} \geq \frac{1}{4\sqrt{\delta}}.$$

Finally

$$\begin{aligned}\rho(\lfloor x\rfloor, \lfloor y\rfloor + 1) &= \ln\frac{\lfloor y\rfloor + 1}{\lfloor x\rfloor} \leq \ln\frac{y + 1}{x - 1} = \ln\frac{y}{x} + \ln\frac{y + 1}{y} + \ln\frac{x}{x - 1}\\ &\leq \delta + \sqrt{\delta} + 4\sqrt{\delta} \leq 6\sqrt{\delta}.\end{aligned}$$

Applying (2.5) we conclude that if $y - x > \sqrt{\delta}$, then

$$|f(x) - f(y)| \leq \omega_{\rho,\sigma}(6\sqrt{\delta}). \tag{2.9}$$

Combining both cases $y - x < \sqrt{\delta}$ and $y - x \geq \sqrt{\delta}$, we obtain from (2.8) and (2.9) that

$$|f(x) - f(y)| \leq \max\bigl(\sqrt{\delta}\,\omega_{\rho,\sigma}(1),\ 2\omega_{\rho,\sigma}(6\sqrt{\delta})\bigr),$$

which implies (2.7). Inequality (2.7) guarantees that $\lim_{\delta\to 0}\Omega_{\rho,f}(\delta) = 0$. ☐

Remark 2.4. Theorem 2.3 was stated for the algebras of *bounded* very slowly oscillating sequences and functions, but the proof of (2.7) does not use the condition of boundedness. Therefore a result analogous to $\mathrm{VSO}(\mathbb{N}) = \{R(f)\colon\ f \in \mathrm{VSO}(\mathbb{R}_+)\}$ holds also for the corresponding classes of sequences and functions without the condition of boundedness.

We finish this section with a brief description of the algebras $\mathrm{VSO}(\mathbb{R}_+)$ and $\mathrm{VSO}(\mathbb{N})$ via their respective compact sets $M(\mathbb{R}_+)$ and $M(\mathbb{N})$ of maximal ideals (multiplicative functionals). We leave details and proofs to the interested reader.

First of all, the following disjoint union representations hold:

$$M(\mathbb{R}_+) = M_0(\mathbb{R}_+) \cup \mathbb{R}_+ \cup M_\infty(\mathbb{R}_+) \quad \text{and} \quad M(\mathbb{N}) = \mathbb{N} \cup M_\infty(\mathbb{N}),$$

where the points of $\mathbb{R}_+$ and $\mathbb{N}$ are identified with the corresponding point evaluation functionals, and the compact sets $M_0(\mathbb{R}_+)$, $M_\infty(\mathbb{R}_+)$, and $M_\infty(\mathbb{N})$ are defined as

$$\begin{aligned} M_0(\mathbb{R}_+) &= \{\varphi \in M(\mathbb{R}_+) : \varphi(f) = 0 \text{ for } f \in \mathrm{VSO}(\mathbb{R}_+) \text{ with } \lim_{x\to 0} f(x) = 0\}, \\ M_\infty(\mathbb{R}_+) &= \{\varphi \in M(\mathbb{R}_+) : \varphi(f) = 0 \text{ for } f \in \mathrm{VSO}(\mathbb{R}_+) \text{ with } \lim_{x\to\infty} f(x) = 0\}, \\ M_\infty(\mathbb{N}) &= \{\varphi \in M(\mathbb{N}) : \varphi(\sigma) = 0 \text{ for } \sigma \in \mathrm{VSO}(\mathbb{N}) \text{ with } \lim_{j\to\infty} \sigma_j = 0\}. \end{aligned}$$

The sets $M_0(\mathbb{R}_+)$ and $M_\infty(\mathbb{R}_+)$ can be identified via the following homeomorphism of $M(\mathbb{R}_+)$:

$$[\widehat{J}(\varphi)](f) = \varphi(J(f)), \quad \text{with} \quad [J(f)](x) = f(1/x).$$

It is easy to check that if a function $f \in \mathrm{VSO}(\mathbb{R}_+)$ satisfies the condition $f(j) = 0$ for all $j \in \mathbb{N}$, then $\lim_{x\to\infty} f(x) = 0$. This implies in turn that each point of $M_\infty(\mathbb{R}_+)$ can be reached by a $\mathbb{N}$-valued net. Such nets thus identify the points of $M_\infty(\mathbb{R}_+)$ and $M_\infty(\mathbb{N})$.

3. From vertical to radial case

For the reader's convenience we recall briefly the results of [8, 9] for the vertical Toeplitz operators.

We denote by Γ_λ the set of all spectral functions (1.2) for $b \in L_\infty(\mathbb{R}_+)$.

Theorem 3.1. *Γ_λ is a dense subset of* $\mathrm{VSO}(\mathbb{R}_+)$.

Proof. We prove first that for each $b \in L_\infty(\mathbb{R}_+)$ the corresponding function $\gamma_{b,\lambda}$ belongs to $\mathrm{VSO}(\mathbb{R}_+)$. The proof given here is different from the original one of [9].

Separating the real and imaginary parts of a symbol it is sufficient to consider the case of the real-valued b. For every $0 < x < y$, by Cauchy's mean value theorem, there exists $\zeta \in (x, y)$ such that

$$\frac{\gamma_{b,\lambda}(x) - \gamma_{b,\lambda}(y)}{\ln(x) - \ln(y)} = \frac{\gamma'_{b,\lambda}(\zeta)}{1/\zeta}. \tag{3.1}$$

Let us estimate from above the derivative of $\gamma_{b,\lambda}$:

$$\begin{aligned}|\gamma'_{b,\lambda}(\zeta)| &= \left|\frac{(\lambda+1)x^\lambda}{\Gamma(\lambda+1)}\int_0^\infty b(v)v^\lambda e^{-\zeta v}dv - \frac{\lambda+1}{\Gamma(\lambda+1)}\int_0^\infty b(v)v^{\lambda+1}e^{-\zeta v}dv\right| \\ &= \left|\frac{(\lambda+1)}{\zeta}\gamma_{b,\lambda}(\zeta) - \frac{(\lambda+1)}{\zeta}\gamma_{b,\lambda+1}(\zeta)\right| \\ &\le \frac{(\lambda+1)2\|b\|_\infty}{\zeta}.\end{aligned}$$

Combining this upper estimate with equation (3.1) we obtain

$$|\gamma_{b,\lambda}(x) - \gamma_{b,\lambda}(y)| \le 2(\lambda+1)\|b\|_\infty |\ln(x) - \ln(y)|,$$

which means that the function $\gamma_{b,\lambda}$ is Lipschitz continuous with respect to ρ. Thus Γ_λ is a subset of $\mathrm{VSO}(\mathbb{R}_+)$.

Recall that $\mathbb{R}_+$ is a locally compact group; dx/x is a Haar measure on $\mathbb{R}_+$, and the uniform structure of $\mathbb{R}_+$ can be induced by the logarithmic metric ρ. Given two functions $f \in L_\infty(\mathbb{R}_+)$ and $g \in L_1(\mathbb{R}_+)$, their Mellin convolution (multiplicative convolution) is defined by

$$(f \star g)(x) = \int_0^\infty f(y)\, g\left(\frac{x}{y}\right)\frac{dy}{y}, \quad x \in \mathbb{R}_+.$$

Given a function $f\colon \mathbb{R}_+ \to \mathbb{C}$, set $\widetilde{f}(x) = f(1/x)$. Then the mapping $f \mapsto \widetilde{f}$ is an involution, and $\widetilde{f \star g} = \widetilde{f} \star \widetilde{g}$, for every $f \in L_\infty(\mathbb{R}_+)$ and $g \in L_1(\mathbb{R}_+)$.

For a generating symbol $b \in L_\infty(\mathbb{R}_+)$, the function $\gamma_{b,\lambda}$ can be written in the form of a Mellin convolution as follows:

$$\gamma_{b,\lambda}(x) = \frac{1}{\Gamma(\lambda+1)}\int_0^\infty b(v)(2xv)^{\lambda+1}\,\mathrm{e}^{-2xv}\,\frac{dv}{v} = \left(\widetilde{b} \star K_\lambda\right)(x),$$

where the function K_λ is defined on $\mathbb{R}_+$ by

$$K_\lambda(z) = \frac{(2z)^{\lambda+1}\,\mathrm{e}^{-2z}}{\Gamma(\lambda+1)}.$$

It is a general fact that the convolution of an integrable function (on a locally compact group) with a bounded function is a uniformly continuous function; in our settings it means that, for every function γ from the set Γ_λ, the composition $\gamma \circ \exp$ is uniformly continuous on $\mathbb{R}$, i.e. $\gamma \in \mathrm{VSO}(\mathbb{R}_+)$. Moreover, as was already proved, every function from Γ_λ is Lipschitz continuous with respect to the metric ρ.

In order to prove the density we note that the sequence of functions

$$\psi_{k,\lambda}(x) = \frac{1}{\mathrm{B}(k+\lambda, k+\lambda)}\,\frac{x^{k+\lambda}}{(1+x)^{2(k+\lambda)}},$$

where B is the Euler Beta function, is a Dirac sequence on the multiplicative group $\mathbb{R}_+$ (see [8, Proposition 5.1] for the non-weighted case; the proof for the weighted case is quite similar). Moreover, every $\psi_{k,\lambda}$ can be represented as $\widetilde{\varphi_{k,\lambda}} \star K_\lambda$, where

$$\varphi_{k,\lambda}(x) = \frac{\Gamma(\lambda+1)\,(2x)^{-\lambda}}{(\Gamma(n+\lambda))^2}\,\frac{d^{n-1}}{dx^{n-1}}\left(\mathrm{e}^{-x}x^{2(n+\lambda)-1}\right);$$

it is easy to see that $\varphi_{k,\lambda} \in L_1(\mathbb{R}_+, dx/x)$.

Now we are ready to finish the proof of density. Given a function $\gamma \in \mathrm{VSO}(\mathbb{R}_+)$, we define a sequence of functions $b_{k,\lambda} = \widetilde{\gamma} \star \varphi_{k,\lambda}$. Since $\varphi_{k,\lambda} \in L_1(\mathbb{R}_+, dx/x)$, we have that $b_{k,\lambda} \in L_\infty(\mathbb{R}_+)$, thus $\gamma_{b_{k,\lambda}} \in \Gamma_\lambda$ and

$$\widetilde{b_{n,\lambda}} \star K_\lambda = \gamma \star \widetilde{\varphi_{k,\lambda}} \star K_\lambda = \gamma \star \psi_{k,\lambda}.$$

Finally, since γ is uniformly continuous on the locally compact group $\mathbb{R}_+$ and the functions $\psi_{k,\lambda}$ form a Dirac sequence, $\gamma \star \psi_{k,\lambda}$ tends to γ uniformly on $\mathbb{R}_+$ as $k \to \infty$. ☐

Passing to the radial case, we observe that in the non-weighted one-dimensional case ($\lambda = 0$, $n = 1$) the sequence $\beta_{a,0}$ is just the restriction to $\mathbb{N}$ of the function $\gamma_{b,0}$, where a and b are related by $a(r) = b(-\ln(r))$. In the weighted case the situation is a bit more complicated: in addition to the variable change $v = -\ln(r)$, two "correcting factors", an inner factor ξ_λ and an outer factor $\eta_{n,\lambda}$ are needed.

Lemma 3.2. *Let $b \in L_\infty(\mathbb{R}_+)$. Define*

$$a(\sqrt{r}) = \xi_{n,\lambda}(r)\,b\left(\frac{-\ln(r)}{2}\right), \qquad 0 < r < 1, \tag{3.2}$$

where

$$\xi_{n,\lambda}(r) = \left(\frac{-\ln(r)}{1-r}\right)^{\lambda}\frac{1}{r^{n-1}}, \qquad 0 < r < 1. \tag{3.3}$$

Then

$$\beta_{a,n,\lambda}(k) = \eta_{n,\lambda}(k)\gamma_{b,\lambda}(k), \qquad k \in \mathbb{N}, \tag{3.4}$$

where

$$\eta_{n,\lambda}(k) = \frac{\Gamma(k+n+\lambda)}{k^{\lambda+1}\,\Gamma(k+n-1)}. \tag{3.5}$$

Proof. Direct computation. We start with (1.1), substitute (3.2), and make change of variables $v = -\ln(r)$:

$$\begin{aligned}\beta_{a,n,\lambda}(k) &= \frac{1}{\mathrm{B}(k+n-1,\lambda+1)}\int_0^1 a(\sqrt{r})r^{k+n-2}(1-r)^\lambda\,dr\\ &= \frac{\Gamma(k+n+\lambda)}{\Gamma(k+n-1)\Gamma(\lambda+1)}\int_0^1 b\left(-\frac{\ln(r)}{2}\right)(-\ln(r))^\lambda r^{k-1}\,dr\\ &= \frac{\Gamma(k+n+\lambda)}{k^{\lambda+1}\,\Gamma(k+n-1)}\,\frac{k^{\lambda+1}}{\Gamma(\lambda+1)}\int_{\mathbb{R}_+} b\left(\frac{v}{2}\right)v^\lambda\,\mathrm{e}^{-kv}\,dv\\ &= \eta_{n,\lambda}(k)\gamma_{b,\lambda}(k).\end{aligned}$$

☐

Note that the function a defined by (3.2) can be unbounded, in general.

Lemma 3.3. *Let $b \in L_\infty(\mathbb{R}_+)$. For every $L > 0$ denote by $\chi_{(0,L)}$ the characteristic function of $(0, L)$. Then*

$$\lim_{L\to+\infty} \sup_{x\geq 1} |\gamma_{b,\lambda}(x) - \gamma_{b\chi_{(0,L)},\lambda}(x)| = 0.$$

Proof. For every $x \geq 1$,

$$|\gamma_{b,\lambda}(x) - \gamma_{b\chi_{(0,L)},\lambda}(x)| \leq \frac{\|b\|_\infty\, x^{\lambda+1}}{\Gamma(\lambda+1)} \int_L^{+\infty} \mathrm{e}^{-xv}\, v^\lambda\, dv$$
$$= \frac{\|b\|_\infty}{\Gamma(\lambda+1)} \int_{Lx}^{+\infty} \mathrm{e}^{-t}\, t^\lambda\, dt \leq \frac{\|b\|_\infty}{\Gamma(\lambda+1)} \int_L^{+\infty} \mathrm{e}^{-t}\, t^\lambda\, dt.$$

The integrability of the function $t \mapsto \mathrm{e}^{-t}\, t^\lambda$ ensures that the last expression tends to 0 as $L \to +\infty$. □

Lemma 3.4. *The sequence $\eta_{n,\lambda} = (\eta_{n,\lambda}(k))_{k\in\mathbb{N}}$ defined by (3.5) tends to 1 as $k \to \infty$, and, in particular, it is bounded.*

Proof. We write

$$\eta_{n,\lambda}(k) = \left(\frac{k+n-1}{k}\right)^{\lambda+1} \frac{\Gamma(k+n-1+\lambda+1)}{\Gamma(k+n-1)(k+n-1)^{\lambda+1}},$$

then using [3, Formula 8.328.2] we obtain required

$$\lim_{k\to\infty} \eta_{n,\lambda}(k) = 1.$$ □

As was already proved, the set Γ_λ is dense in $\mathrm{VSO}(\mathbb{R}_+)$. Now we are going to deduce from this fact that $\mathrm{B}_{n,\lambda}$ is a dense subset of $\mathrm{VSO}(\mathbb{N})$.

Theorem 3.5. *For each $a \in L_\infty([0, 1))$, $\beta_{a,n,\lambda}$ belongs to* $\mathrm{VSO}(\mathbb{N})$.

Proof. We start from a function $a \in L_\infty([0, 1))$, and introduce $a_1 = a \cdot \chi_{[\frac{1}{2},1)}$ and $a_2 = a - a_1 = a \cdot \chi_{[0,\frac{1}{2})}$. We have that $\beta_{a_2,n,\lambda} \in c_0 \subset \mathrm{VSO}(\mathbb{N})$. Thus it is sufficient to show that $\beta_{a_1,n,\lambda} \in \mathrm{VSO}(\mathbb{N})$. Inverting (3.2) we define

$$b(\tfrac{v}{2}) \colon = a_1 \left(e^{-\frac{v}{2}}\right) \left(\frac{1-e^{-v}}{v}\right)^\lambda e^{-v(n-1)}.$$

As $\lim_{v\to 0}(1 - e^{-v})/v = 1$, the function b is bounded, thus $\gamma_{b,\lambda} \in \mathrm{VSO}(\mathbb{R}_+)$ and $\gamma_{b,\lambda}|_{\mathbb{N}} \in \mathrm{VSO}(\mathbb{N})$.

By (3.4), we have $\beta_{a_1,n,\lambda}(k) = \eta_{n,\lambda}(k)\gamma_{b,\lambda}(k)$, $k \in \mathbb{N}$, and thus $\beta_{a_1,n,\lambda} \in \mathrm{VSO}(\mathbb{N})$ as a product of two $\mathrm{VSO}(\mathbb{N})$-sequences. □

Theorem 3.6. *The set $\mathrm{B}_{n,\lambda}$ is dense in* $\mathrm{VSO}(\mathbb{N})$.

Proof. We start from a sequence $\nu \in \mathrm{VSO}(\mathbb{N})$ and define the sequence σ as

$$\sigma(k) := \frac{\nu(k)}{\eta_{n,\lambda}(k)}, \qquad k \in \mathbb{N}.$$

By Lemma 3.4, $\sigma \in \mathrm{VSO}(\mathbb{N})$. Using Theorem 2.3 we construct a function f in $\mathrm{VSO}(\mathbb{R}_+)$ such that σ is the restriction of f to $\mathbb{N}$. Since $f \in \mathrm{VSO}(\mathbb{R}_+)$, by Theorem 3.1, for each $\varepsilon > 0$ there exists $g \in L_\infty(\mathbb{R}_+)$ such that

$$\|f - \gamma_{g,\lambda}\|_\infty < \frac{\varepsilon}{2\|\eta_{n,\lambda}\|_\infty}.$$

By Lemma 3.3, we take $L > 0$ such that

$$\sup_{x\geq 1} |\gamma_{g,\lambda}(x) - \gamma_{g\chi_{(0,L)},\lambda}(x)| < \frac{\varepsilon}{2\|\eta_{n,\lambda}\|_\infty}.$$

Define

$$a(\sqrt{r}) = \chi_{(0,L)}\left(\frac{-\ln r}{2}\right)\xi_{n,\lambda}(r) g\left(\frac{-\ln r}{2}\right), \qquad 0 < r < 1.$$

The factor $\chi_{(0,L)}$ insures that the function a vanishes near zero and is bounded. By Lemma 3.2

$$\beta_{a,n,\lambda}(k) = \eta_{n,\lambda}(k)\gamma_{g\chi_{(0,L)},\lambda}(k).$$

Therefore for every $k \in \mathbb{N}$

$$\begin{aligned} |\nu(k) - \beta_{a,n,\lambda}(k)| &= \eta_{n,\lambda}(k)|\sigma(k) - \gamma_{g\chi_{(0,L)},\lambda}(k)| \\ &\leq \|\eta_{n,\lambda}\|_\infty \left(\|f - \gamma_{g,\lambda}\|_\infty + \sup_{x\geq 1}|\gamma_{g,\lambda}(x) - \gamma_{g\chi_{(0,L)},\lambda}(x)|\right) < \varepsilon. \end{aligned}$$ □

Corollary 3.7. *For every $n \in \mathbb{N}$ and $\lambda > -1$ the C^*-algebra generated by Toeplitz operators $T_{a,n,\lambda}$ with bounded measurable radial symbols a is isometrically isomorphic to the algebra* VSO($\mathbb{N}$)*. The isomorphism is generated by the assignment* $T_{a,n,\lambda} \longmapsto \beta_{a,n,\lambda}$.

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