One Szegő–Widom Limit Theorem

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This file was prepared by Egor A. Maximenko in 2014. I am grateful to my PhD advisor Igor B. Simonenko (1935–2008) and to Vladimir A. Vasil'ev for this publication. The main idea of this paper was not so new as we thought, see MathReview 2107785.

Szegő and Widom have obtained a formula similar to formula (1) below but with a remainder term of the form $o(1)$ (see [4, Chapter 5]). However, it has not been shown up till now that, for a sufficiently smooth symbol σ, the principal part of that formula gives a complete expansion of $T_f A_N$ in powers of $\frac{1}{N}$, see (2). The results presented here fill this gap in both discrete and continuous cases. They predict that, based on formulas (1)–(4), algorithms for evaluating the generalized traces of $T_f A_N$ and $\mathfrak{T}_f A_N$ should be much more efficient.

Let $\mathbb{C}$ be the field of complex numbers, $\mathbb{T} = \{z \mid z \in \mathbb{C},\ |z| = 1\}$; s be the Lebesgue measure, i.e., the length on $\mathbb{T}$, and $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and positive integers, respectively. Denote by l_k ($k \in \mathbb{Z}$) the linear functional on $L_1(\mathbb{T})$ defined by the equality

$$l_k\phi = \frac{1}{2\pi}\int\limits_{\mathbb{T}} t^{-k}\phi(t)\,ds,$$

and let K_m ($m \in \mathbb{N}$) be the space of continuous functions ϕ defined on $\mathbb{T}$ and obeying the conditions

$$\sum_{k\in\mathbb{Z}} |k|^m |l_k\phi|^2 < +\infty, \qquad \sum_{k\in\mathbb{Z}} |l_k\phi| < +\infty.$$

Let us introduce some notation.

C_n ($n \in \mathbb{N}$) is the set of square matrix-valued functions of order n that are continuous on S. Z_n is the set of matrix-valued functions in C_n that (a) are everywhere nonsingular and (b) have zero left and right partial indices[1] (see [7]). Note that Z_1 coincides with the set of continuous nonvanishing functions on S whose argument has a zero increment while traversing S.

[1]The term "partial indices" was introduced by Muskhelishvili and Vekua [6, 5]. However, we will distinguish two types of partial indices (left and right), as was done in [7].

$G(\sigma)$, with $\sigma \in C_n$, is the set of those $\lambda \in \mathbb{C}$ for which $\lambda E - \sigma \in Z_n$, where E is an identity matrix, $F(\sigma) = \mathbb{C} \setminus G(\sigma)$.

$\mathrm{tr}(\sigma)$ is the trace of a square numerial matrix A.

$T_f A$, where f is a function of a complex variable and A is a square numerical matrix, is the sum $\sum_{\lambda\in\Lambda} \rho(\lambda) f(\lambda)$, where Λ is the set of eigenvalues of A that belong to the domain of f and $\rho(\lambda)$ is their multiplicity.

Theorem 1. *Suppose that $\sigma \in C_n$; f is an analytic function of a complex variable defined on an open set containing $F(\sigma)$; $l_k\sigma$ ($k \in \mathbb{Z}$) is the numerical square matrix of order n defined by the equality $l_k\sigma = (l_k\sigma_{i,j})$, where $\sigma_{i,j}$ ($i, j \in \{1, 2, \ldots, n\}$) are the elements of the matrix σ. Denote by A_N ($N \in \mathbb{N}$) a block matrix $(a_{i,j})$ ($i, j \in \{1, 2, \ldots, N\}$) such that $a_{i,j} = l_{i-j}\sigma$. Then the following is true:*

(i) *if $m \in \mathbb{N}$ and the elements of σ belong to K_m, then, as $N \to +\infty$, we have the asymptotic formula*

$$T_f A_N = c_0 N + c_1 + o(N^{1-m}). \tag{1}$$

Here $c_0, c_1 \in \mathbb{C}$ and c_0 is calculated by the formula

$$c_0 = \frac{1}{2\pi} \int_S \psi \, ds,$$

where ψ is the function on S defined by the equality

$$\psi(t) = \mathrm{tr}[f(\sigma(t))];$$

(ii) *if the elements of σ are infinitely differentiable, then*

$$T_f A_N = c_0 N + c_1 + O(N^{-\infty}). \tag{2}$$

Here and below, this means that the remainder term of a formula decreases faster than any negative power.

Note that, for every $t \in \mathbb{T}$, the spectrum of $\sigma(t)$ is contained in the domain of f. Consequently, the matrix $f(\sigma(t))$ is defined for every $t \in \mathbb{T}$.

Let us turn to the continuous case.

Suppose that $\mathbb{R}$ is the field of real numbers, $\dot{\mathbb{R}}$ is the real line extended by one point at infinity ∞; $\Phi\varphi$, with $\varphi \in L_1(\mathbb{R})$, is the function on $\mathbb{R}$ defined by the equality

$$(\Phi\varphi)(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(t) \exp(-itx) \, dt,$$

where i is the imaginary unit; and $\mathfrak{K}_m$ ($m \in \mathbb{N}$) is the set of functions $\varphi \in L_1(\mathbb{R})$ such that

$$\Phi\varphi \in L_1(\mathbb{R}) \quad \text{and} \quad \int_{-\infty}^{+\infty} |(\Phi\varphi)(x)|^2 \, |x|^m \, dx < +\infty.$$

Note that, if $\varphi \in \mathfrak{K}_m$, then φ is continuous, continuously extendable to $\dot{\mathbb{R}}$, and $\varphi(\infty) = 0$.

We introduce the necessary notation.

$\mathfrak{C}_n$ ($n \in \mathbb{N}$) is the set of square matrix-valued functions M of order n in $\mathbb{R}$ that are representable as $M = M_0 + M_1$, where M_0 is a numerical matrix and M_1 is a matrix-valued function whose elements can be represented by absolutely convergent Fourier integrals. $\mathfrak{Z}_n$ is the set of matrix-valued functions from $\mathfrak{C}_n$ that are everywhere nonsingular (including the point at infinity, to which they are continuously extendable) and whose left and right partial indices are zero [8, p. 33].

$\mathfrak{G}(\sigma)$ ($\sigma \in \mathfrak{C}_n$) is the set of $\lambda \in \mathbb{C}$ for which $\lambda E - \sigma \in \mathfrak{Z}_n$, where E is an identity matrix; $\mathfrak{F}(\sigma) = \mathbb{C} \setminus \mathfrak{G}(\sigma)$.

$\mathfrak{U}$ is the set of analytic functions f of a complex variable defined on an open subset of the complex plane with zero and such that $f(0) = 0$.

$\mathfrak{T}_f A$, where $f \in \mathfrak{U}$ and A is a nuclear operator (trace-class operator), is the sum $\sum_{\lambda \in \Lambda} \rho(\lambda) f(\lambda)$, where Λ is the set of nonzero eigenvalues of A that belong to the domain of f and $\rho(\lambda)$ is their multiplicity. Note that this sum makes sense because A is nuclear and $f(0) = 0$.

Theorem 2. *Suppose that $\sigma \in \mathfrak{C}_n$; the elements of σ belong to $L_1(\mathbb{R})$; $f \in \mathfrak{U}$ is a function whose domain contains $\mathfrak{F}(\sigma)$; k is the square matrix-valued function of order n defined in $\mathbb{R}$ by the equality $k = (\Phi(\sigma_{i,j}))$ ($i, j \in \{1, 2, \ldots, n\}$), where $\sigma_{i,j}$ are the elements of σ; and A_N ($0 < N < +\infty$) is the operator in $L_1^n([0, N])$ acting by the rule*

$$(A_N\varphi)(y) = \int_0^N k(y - x)\varphi(x) \, dx.$$

Then the following is true:

(i) *If $m \in \mathbb{N}$ and the elements of σ belong to $\mathfrak{K}_m$, then, as $N \to +\infty$, we have the asymptotic formula*

$$\mathfrak{T}_f A_N = c_0 N + c_1 + o(N^{1-m}). \tag{3}$$

Here $c_0, c_1 \in \mathbb{C}$, and

$$c_0 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(t) \, dt,$$

where ψ is the function on $\mathbb{R}$ defined by the equality

$$\psi(t) = \operatorname{tr}[f(\sigma(t))].$$

(ii) *If the elements of σ are infinitely differentiable functions representable, together with all of their derivatives, by absolutely summable Fourier integrals, then*

$$\mathfrak{T}_f A_N = c_0 N + c_1 + O(N^{-\infty}). \tag{4}$$

Note that, in this case, the matrix $f(\sigma(t))$ is also well defined for every $t \in \mathbb{R}$.

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