From convergence in distribution to uniform convergence of quantile functions

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based on joint works with Johan Manuel Bogoya, Albrecht Böttcher, and Sergei M. Grudsky

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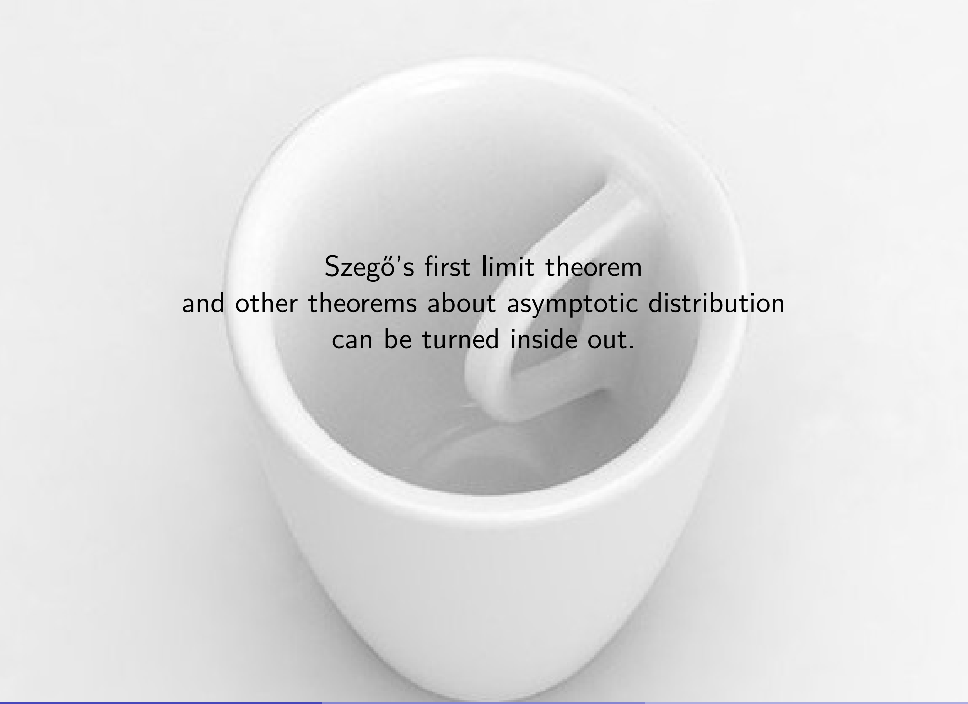
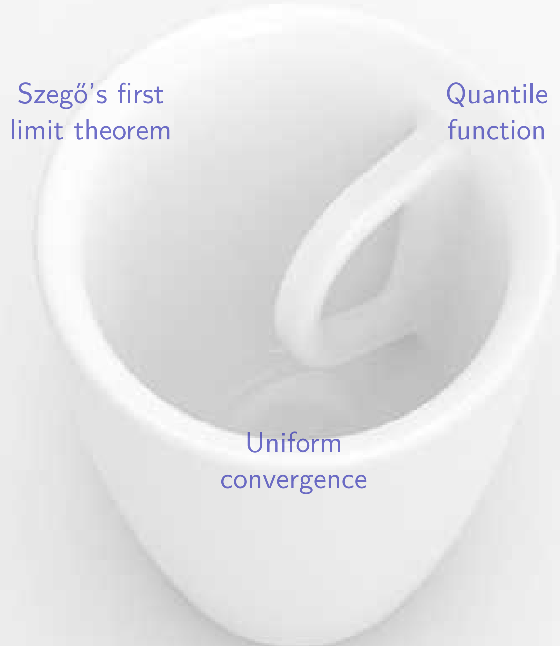
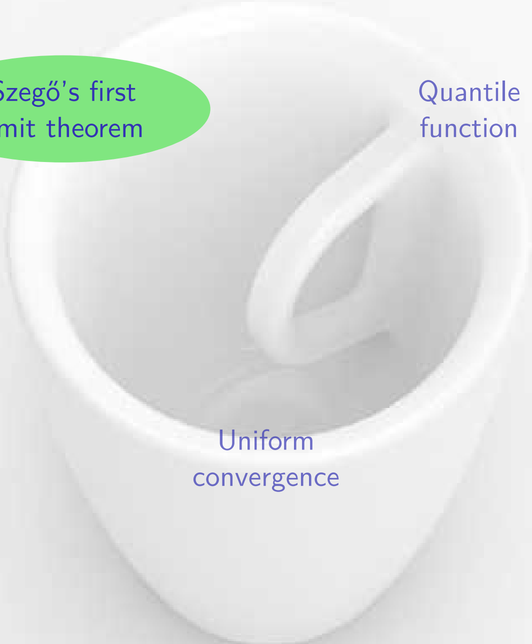
International Workshop
Wiener–Hopf method, Toeplitz operators, and their applications
Boca del Río, Veracruz, México
November, 2015



Szegő's first limit theorem
and other theorems about asymptotic distribution
can be turned inside out.


Szegő's first limit theorem
Quantile function
Uniform convergence



Szegő's first limit theorem

Quantile function

Uniform convergence

Toeplitz matrices

$$T_5(a) = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ a_3 & a_2 & a_1 & a_0 & a_{-1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}.$$

It is convenient to think that a_j are the Fourier coefficients of a certain function a defined on $[0, 2\pi]$:

$$a_j = \frac{1}{2\pi} \int_0^{2\pi} a(\theta)\, e^{-ji\theta}\, d\theta.$$

The function a is called the *generating symbol* of the matrices

$$T_n(a) = \left[a_{j-k}\right]_{j,k=1}^{n}.$$

Hermitian Toeplitz matrices, real bounded symbols

We suppose that the generating symbol is bounded and real:

$$a \in L^\infty([0, 2\pi], \mathbb{R}).$$

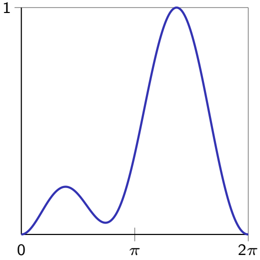
In this case the Toeplitz matrices are Hermitian:

$$a_{-k} = \overline{a_k}, \qquad a_0 \in \mathbb{R}.$$

$$T_5(a) = \begin{bmatrix} a_0 & \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} \\ a_1 & a_0 & \overline{a_1} & \overline{a_2} & \overline{a_3} \\ a_2 & a_1 & a_0 & \overline{a_1} & \overline{a_2} \\ a_3 & a_2 & a_1 & a_0 & \overline{a_1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}.$$

Behavior of the eigenvalues of Hermitian Toeplitz matrices

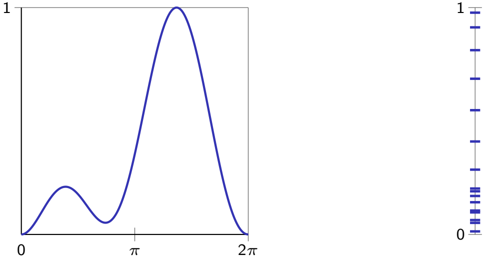
Graph of a



Eigenvalues of $T_8(a)$

Behavior of the eigenvalues of Hermitian Toeplitz matrices

Graph of a Eigenvalues of $T_{16}(a)$



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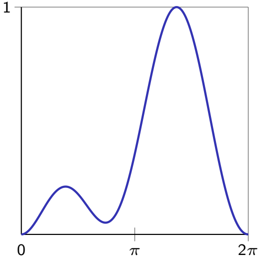
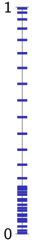


Eigenvalues of $T_{32}(a)$



Behavior of the eigenvalues of Hermitian Toeplitz matrices

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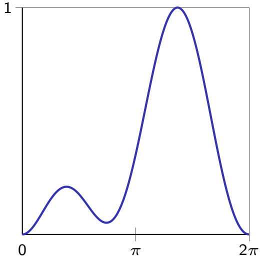

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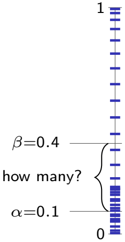
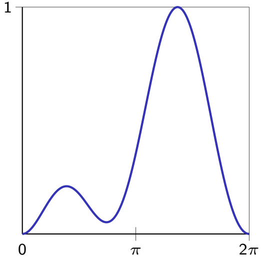


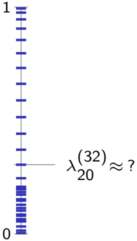
First question: How many eigenvalues are in $[\alpha, \beta]$?

Behavior of the eigenvalues of Hermitian Toeplitz matrices

Graph of a



Eigenvalues of $T_{32}(a)$



First question: How many eigenvalues are in $[\alpha, \beta]$?

Second question: $\lambda_j^{(n)} \approx ?$

Szegő's first limit theorem (1920)

distribution of the eigenvalues of Hermitian Toepltiz matrices

generating symbol
$a \in L^\infty([0,2\pi],\mathbb{R})$

test function
$\varphi \in C(\mathbb{R})$

$$\frac{1}{n}\sum_{j=1}^{n}\varphi(\lambda_j^{(n)}) \longrightarrow \frac{1}{2\pi}\int_0^{2\pi}\varphi(a(\theta))\,\mathrm{d}\theta$$

Another form of the Szegő's first limit theorem

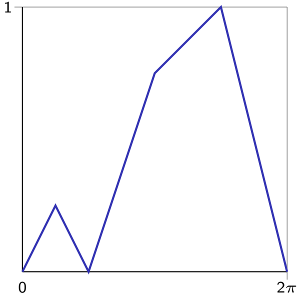
generating symbol
$a \in L^\infty([0, 2\pi], \mathbb{R})$

$\alpha < \beta$
$a(\theta) \neq \alpha, \beta$ a.e.

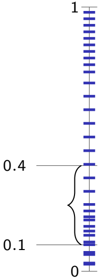
$$\frac{\#\{j\colon\ \alpha \le \lambda_j^{(n)} \le \beta\}}{n} \longrightarrow \frac{\mu_{\mathbb{R}}(a^{-1}([\alpha, \beta]))}{2\pi}$$

Example

Graph of a

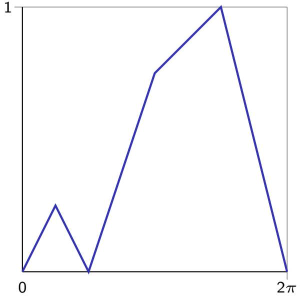


Eigenvalues of $T_{32}(a)$



Example

Graph of a



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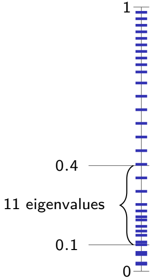
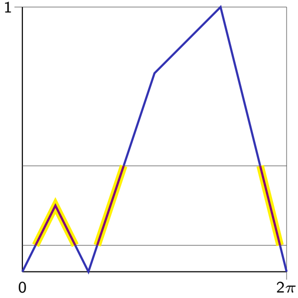


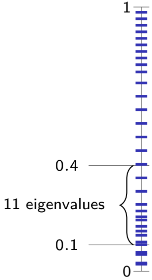
$$\frac{11}{32} \approx 0.344$$

Example

Graph of a



Eigenvalues of $T_{32}(a)$



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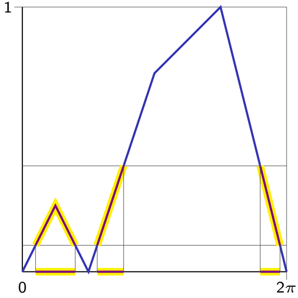
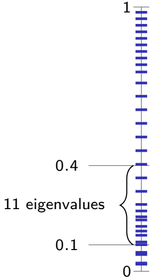


Eigenvalues of $T_{32}(a)$



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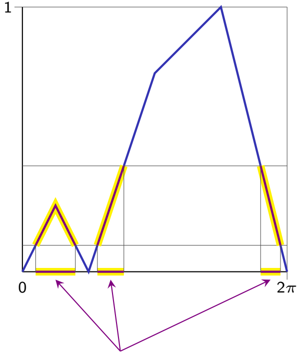
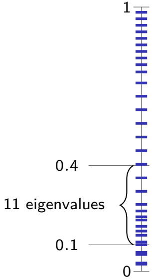


$$\frac{\mu_{\mathbb{R}}\{\theta : \ 0.1 \le a(\theta) \le 0.4\}}{2\pi} = 0.325$$

Eigenvalues of $T_{32}(a)$

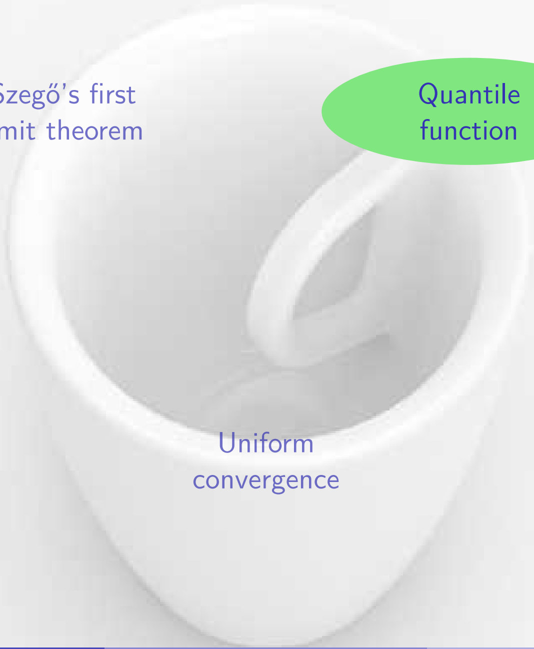


$$\frac{11}{32} \approx 0.344$$

Szegő found an approximate answer to the first question:
how many eigenvalues belong to a given segment $[\alpha, \beta]$?

The second question was still open:

$$\lambda_j^{(n)} \approx ?$$


Szegő's first limit theorem
Quantile function
Uniform convergence

Definition of the quantile function

$\mathcal{BPM}(\mathbb{R}) :=$ Borel probability measures over $\mathbb{R}$.

Given $\mu \in \mathcal{BPM}(\mathbb{R})$, one defines:
the cumulative distribution function $F_\mu \colon \mathbb{R} \to [0,1]$,

$$F_\mu(v) := \mu(-\infty, v],$$

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Denote by support of μ:

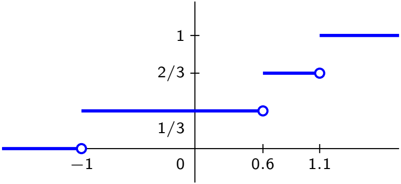
$$\operatorname{supp}(\mu) := \{v \in \mathbb{R}\colon \quad \forall \varepsilon > 0 \quad \mu(v - \varepsilon, v + \varepsilon) > 0\}.$$

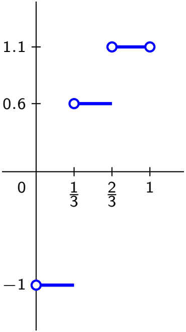
Quantile function associated to a list of numbers

Consider a list of real numbers:

$$X = (-1,\ 0.6,\ 1.1).$$

Using the normalized counting measure
define the cdf and the quantile function:





Quantile function of a list of numbers

118 100 195 166 164 123 102 172 164 117

The same numbers in the ascending order ($\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{10}$):

100 102 117 118 123 164 164 166 172 195

$$Q(1/3) = \alpha_{\lceil 10/3 \rceil} = \alpha_4 = 118.$$

Quantile function associated to a function

Let $a \in L^\infty([0,2\pi],\mathbb{R})$.

Pushforward measure $\mu \in \mathcal{BPM}(\mathbb{R})$:

$$\mu(B) := \frac{1}{2\pi}\mu_{\mathbb{R}}(a^{-1}(B)).$$

$F_a :=$ the cumulative distribution function of a:

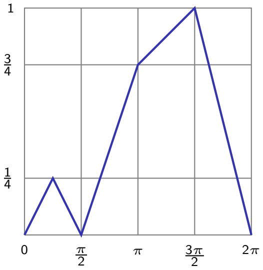
$$F_a(v) := \frac{1}{2\pi}\,\mu_{\mathbb{R}}\,\{\theta \in [0,2\pi]\colon\ a(\theta) \le v\}, \qquad v \in \mathbb{R}.$$

$Q_a :=$ the corresponding quantile function:

$$Q_a(p) := \inf\{v \in \mathbb{R}\colon\ F_a(v) \ge p\}, \qquad p \in (0,1).$$

Construction of the quantile function associated to a piecewise-linear generating symbol

Graph of a



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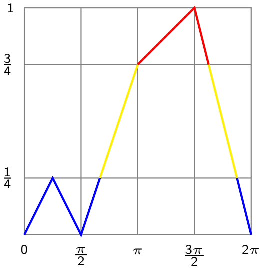

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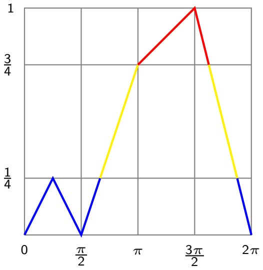
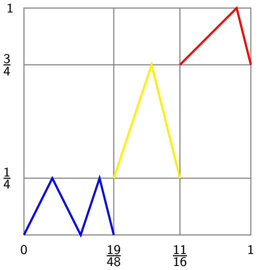
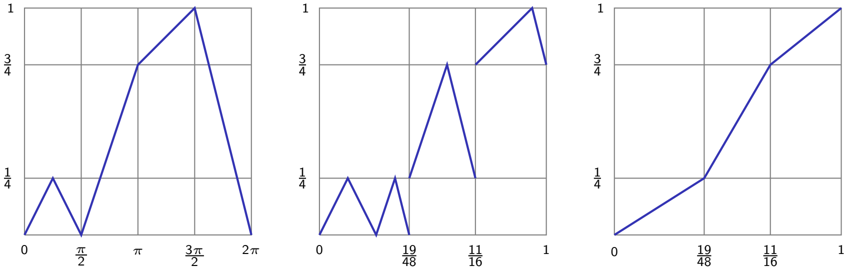
Graph of a





Construction of the quantile function associated to a piecewise-linear generating symbol

Graph of a Graph of Q_a



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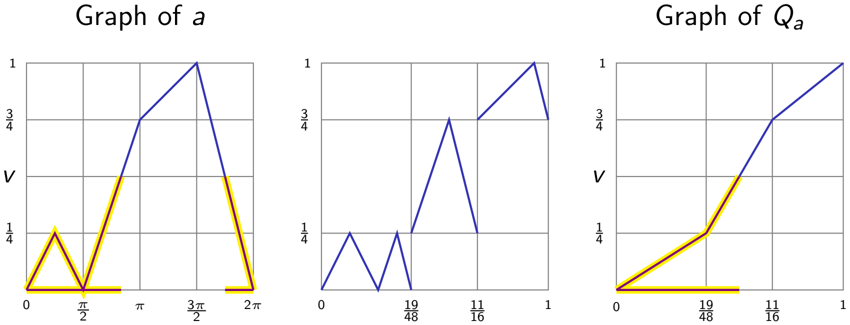
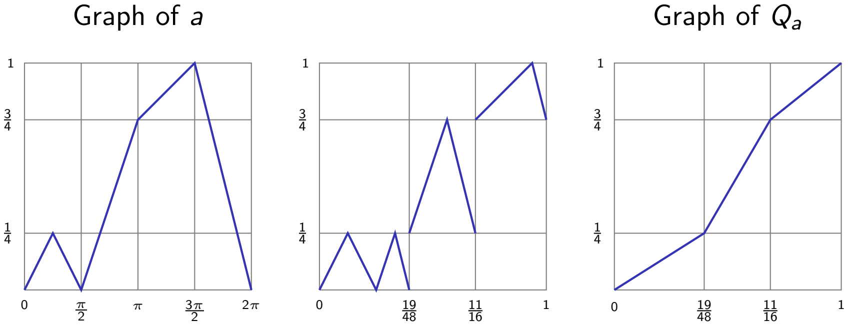


a and Q_a are identically distributed:

$$\frac{1}{2\pi}\mu_{\mathbb{R}}\{\theta \in [0,2\pi]: \ a(\theta) \le v\} = \mu_{\mathbb{R}}\{p \in [0,1]: \ Q_a(p) \le v\}$$

Construction of the quantile function associated to a piecewise-linear generating symbol



$$a \xrightarrow{\text{reordering in Lebesgue-style}} Q_a$$

Continuity of the quantile function

Let $\mu \in \mathcal{BPM}(\mathbb{R})$ with compact $\mathrm{supp}(\mu)$.
Then the following conditions are equivalent:

$\mathrm{supp}(\mu)$ is connected

$\Updownarrow$

F_μ is strictly increasing

$\Updownarrow$

Q_μ is continuous

Convergence in distribution ($\mu_n \rightsquigarrow \Lambda$)

(convergence in law, weak convergence)

Let $\Lambda \in \mathcal{BPM}(\mathbb{R})$ and let $(\mu_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{BPM}(\mathbb{R})$. Then the following conditions are equivalent.

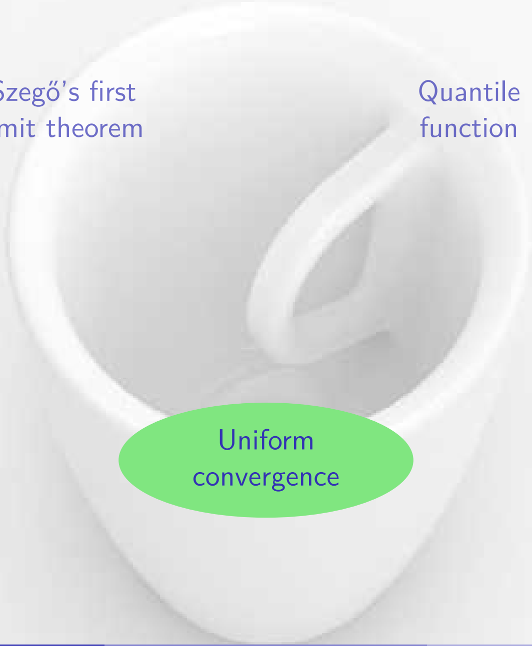
$$\forall \varphi \in C_b(\mathbb{R}) \qquad \lim_{n\to\infty} \int_{\mathbb{R}} \varphi \, d\mu_n = \int_{\mathbb{R}} \varphi \, d\Lambda$$

$$\Updownarrow$$

$$\forall v \in \mathcal{C}(F_\Lambda) \qquad \lim_{n\to\infty} F_{\mu_n}(v) = F_\Lambda(v)$$

$$\Updownarrow$$

$$\forall p \in \mathcal{C}(Q_\Lambda) \qquad \lim_{n\to\infty} Q_{\mu_n}(p) = Q_\Lambda(p)$$


Szegő's first limit theorem
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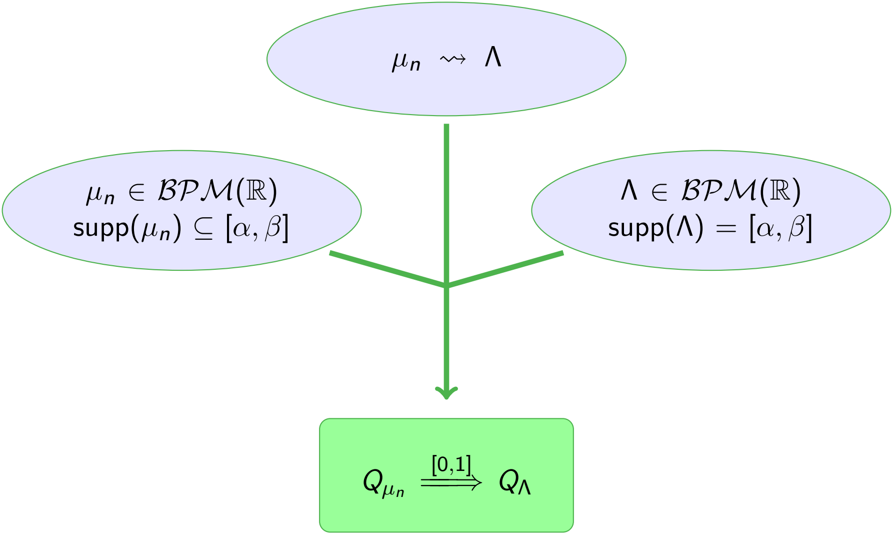
Main result



$\mu_n \rightsquigarrow \Lambda$
$\mu_n \in \mathcal{BPM}(\mathbb{R})$
$\operatorname{supp}(\mu_n) \subseteq [\alpha, \beta]$
$\Lambda \in \mathcal{BPM}(\mathbb{R})$
$\operatorname{supp}(\Lambda) = [\alpha, \beta]$
$Q_{\mu_n} \overset{[0,1]}{\Longrightarrow} Q_\Lambda$

Application to Toeplitz matrices: uniform approximation of the eigenvalues

$a \in L^\infty([0, 2\pi], \mathbb{R})$

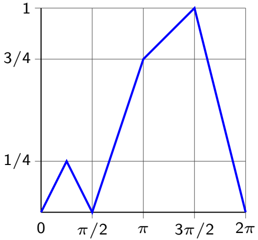
$\mathcal{R}(a) = [\alpha, \beta]$

$$\max_{1 \le j \le n} \left| \lambda_j^{(n)} - Q_a(j/n) \right| \longrightarrow 0$$

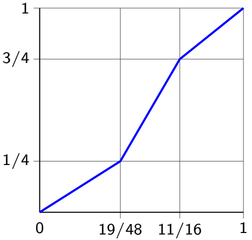
First example

continuous piecewise-linear generating symbol

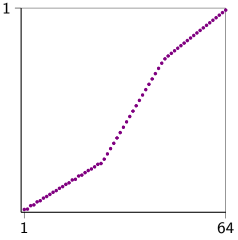
Graph of a



Graph of Q_a



Eigenvalues of $T_{64}(a)$

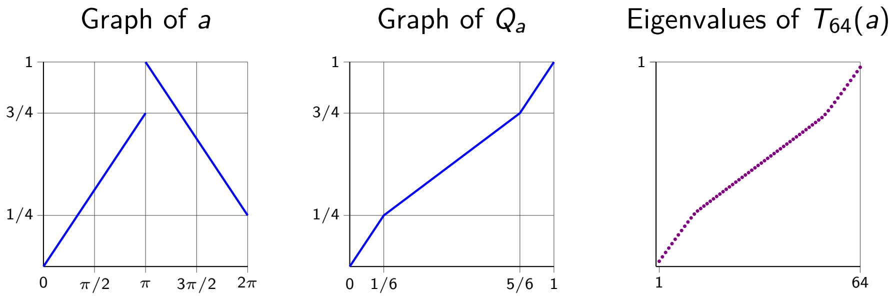


Every eigenvalue $\lambda_j^{(n)}$ is shown as a point $\left(\frac{j}{n},\ \lambda_j^{(n)}\right)$.

The third picture mimics the second one.

Second example

a is not continuous, but $\mathcal{R}(a)$ is connected

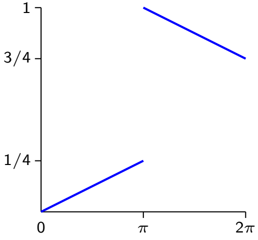


In this example, $\lambda_j^{(n)}$ is also uniformly approximated by $Q_a(j/n)$ as $n \to \infty$.

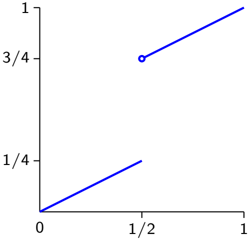
Third example

If $\mathcal{R}(a)$ is not connected, then the uniform convergence fails

Graph of a



Graph of Q_a



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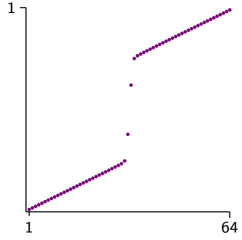
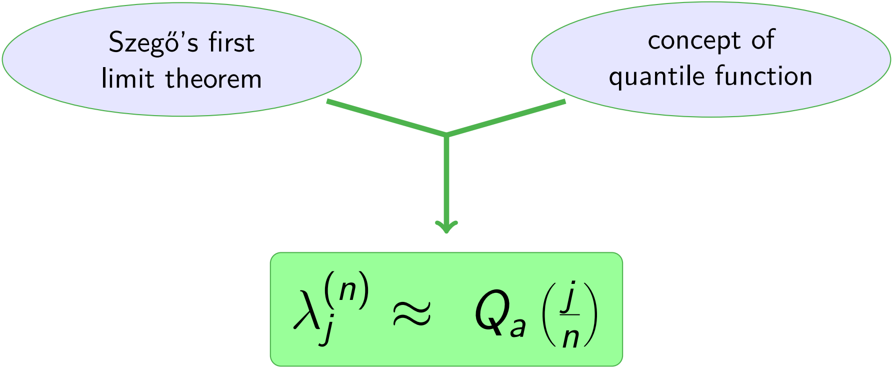
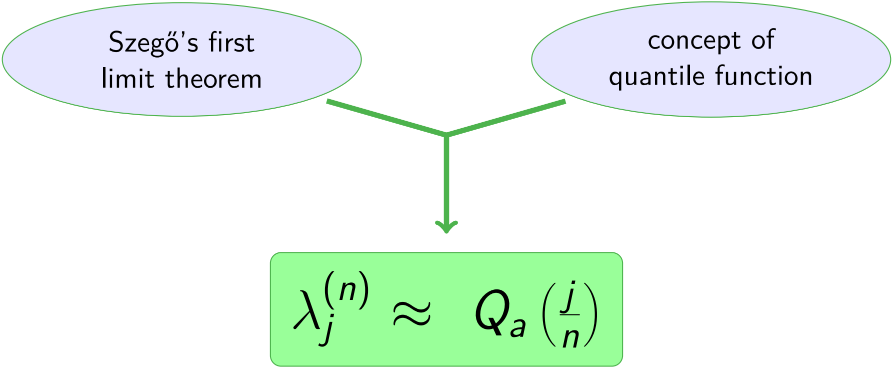


In this example, $\lambda_{\lfloor n/2 \rfloor}^{(n)}$ can not be approximated by values of Q_a.

Summary


Szegő's first limit theorem
concept of quantile function
$\lambda_j^{(n)} \approx Q_a\left(\frac{j}{n}\right)$

Summary



Similar results:

Di Benedetto, Serra, Fiorentino (1993): pointwise convergence.

Trench (2012): convergence in L^1 sence (Riemann-style reordering of a).

More applications

There are many other results about asymptotic distribution:

- Avram–Parter theorem for singular values of Toeplitz matrices.
- Szegő type theorems for locally Toeplitz matrices.
- Lévy's arcsine law for random walks.
- Weyl's theorem about uniformly distributed sequences.

Applying the concept of quantile function
one easily deduces corollaries about uniform approximation.

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Thanks for attention!