

Eigenvalues of radial Toeplitz operators, log-oscillation and sqrt-oscillation

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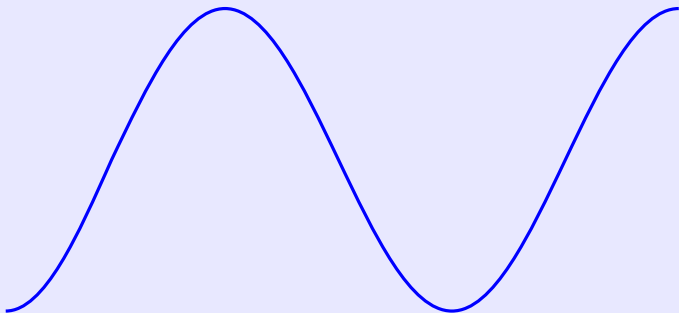
based on joint works with Kevin Esmeral, Sergei Grudsky,
Crispin Herrera Yañez and Nikolai Vasilevski

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Oscillating sequences

Fock case



Approximation

Bergman case

Bounded uniformly continuous functions

Modulus of continuity

Given $f: \mathbb{R} \rightarrow \mathbb{C}$, define $\omega_f: [0, +\infty) \rightarrow [0, +\infty)$ by

$$\omega_f(\delta) := \sup\{|f(x) - f(y)|: x, y \in \mathbb{R}, |x - y| \leq \delta\}.$$

Bounded uniformly continuous functions

$$C_u(\mathbb{R}) := \left\{ f \in C^{\mathbb{R}}: \sup_{x \in \mathbb{R}} |f(x)| < +\infty \quad \wedge \quad \lim_{\delta \rightarrow 0^+} \omega_f(\delta) = 0 \right\}.$$

Proposition. $C_u(\mathbb{R})$ is a C^* -subalgebra of $L^\infty(\mathbb{R})$.

Convolution, Fourier transform

Convolution

Given $k \in L^1(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$,

$$(k * f)(x) := \int_{\mathbb{R}} k(x - y)f(y) dy.$$

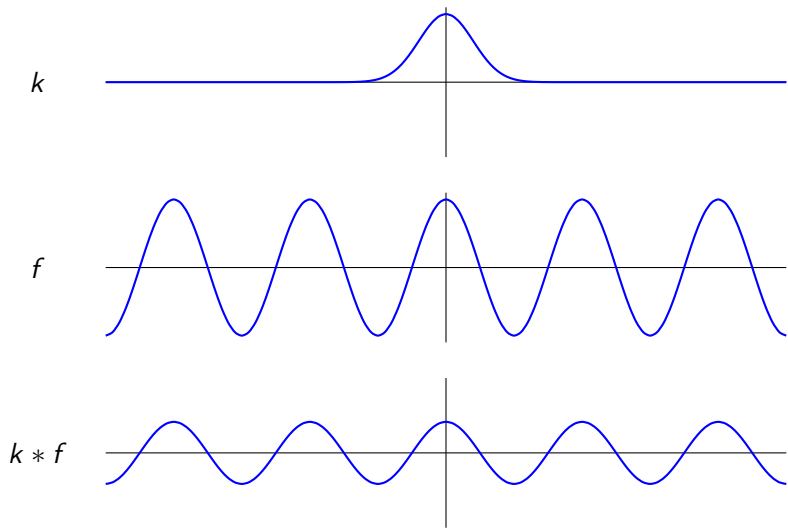
It is easy to see that $k * f \in C_u(\mathbb{R})$.

Fourier transform

Given $k \in L^1(\mathbb{R})$,

$$\widehat{k}(t) := \int_{\mathbb{R}} k(x) e^{-2\pi i xt} dx.$$

Convolution and oscillations: example



Bounded uniformly continuous functions can be uniformly approximated by convolutions

Approximation Theorem. Let $k \in L^1(\mathbb{R})$ satisfy Wiener's condition:

$$\forall t \in \mathbb{R} \quad \widehat{k}(t) \neq 0.$$

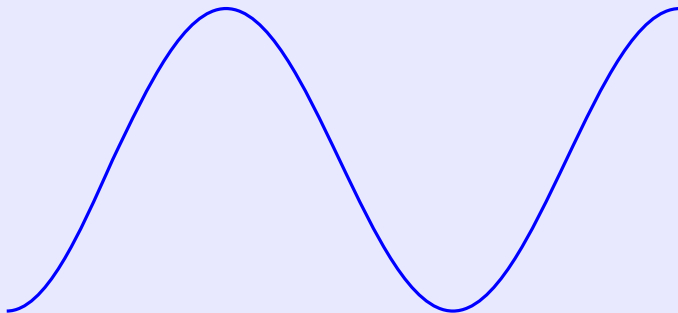
Then $\{k * f : f \in L^\infty(\mathbb{R})\}$ is a dense subset of $C_u(\mathbb{R})$.

We have not found this result in the literature, so we included a proof in Esmeral, Maximenko (2016) doi:10.1007/s11785-016-0557-0

The theorem is very close to Tauberian Wiener's theorem, and our proof is based on the well-known "Wiener's approximate deconvolution technique": Dirac sequences and Wiener's Division Lemma.

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Log-oscillating sequences (Robert Schmidt, 1924)

Definition. LO := bounded sequences $x = (x_n)_{n=0}^\infty$ such that

$$\lim_{\substack{m+1 \\ n+1} \rightarrow 1} |x_m - x_n| = 0.$$

In other words, LO is the set of all bounded functions $\mathbb{N}_0 \rightarrow \mathbb{C}$ that are uniformly continuous with respect to the log-metric:

$$\rho_{\log}(m, n) := |\log(m+1) - \log(n+1)|,$$

$$\omega_{\rho_{\log}, x}(\delta) := \sup\{|x_m - x_n| : m, n \in \mathbb{N}_0, \rho_{\log}(m, n) \leq \delta\},$$

$$\text{LO} := \{x \in \ell^\infty : \lim_{\delta \rightarrow 0^+} \omega_{\rho_{\log}, x}(\delta) = 0\}.$$

Proposition. LO is a C^* -subalgebra of ℓ^∞ .

Log-oscillating sequences can be obtained from bounded uniformly continuous functions

Proposition. The operator $\Phi: C_u(\mathbb{R}) \rightarrow \text{LO}$, defined by

$$(\Phi f)_n := f(\log(n+1)),$$

is an epimorphism of C^* -algebras.

Idea to prove the surjective property:

Given a sequence $x \in \text{LO}$, put

$$f(\log(n+1)) := x_n \quad (n \in \mathbb{N}_0)$$

and apply the linear interpolation between the nodes $\log(n+1)$.

Sqrt-oscillating sequences

RO := the set of all bounded sequences that are uniformly continuous with respect to the sqrt-metric:

$$\rho_{\text{sqrt}}(m, n) := |\sqrt{m} - \sqrt{n}|.$$

Proposition. RO is a C^* -subalgebra of ℓ^∞ .

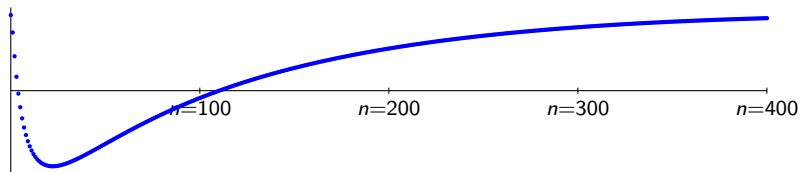
Proposition. The operator $\Psi: C_u(\mathbb{R}) \rightarrow \text{RO}$, defined by

$$(\Psi f)_n := f(\sqrt{n}),$$

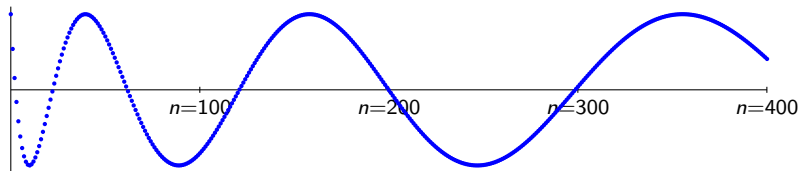
is an epimorphism of C^* -algebras.

Example of log-oscillation and sqrt-oscillation

$$x_n = \cos(\log(n + 1)):$$

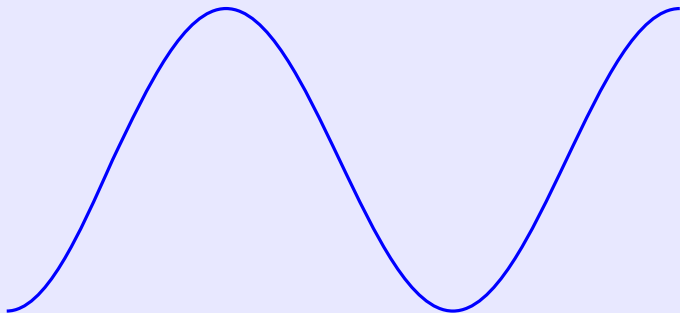


$$x_n = \cos(\sqrt{n}):$$



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Bergman space on the unit disk

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

$$\mu_{\mathbb{D}} := \frac{1}{\pi} \text{ Lebesgue plane measure.}$$

$$\mathcal{A}^2(\mathbb{D}) := \{f \in H(\mathbb{D}) : f \in L^2(\mathbb{D})\}.$$

Bergman projection $P_{\mathbb{D}} :=$ orthogonal projection $L^2(\mathbb{D}) \rightarrow \mathcal{A}^2(\mathbb{D})$.

Orthonormal basis in $\mathcal{A}^2(\mathbb{D})$ consisting of normalized monomials:

$$e_n(z) = \sqrt{n+1} z^n.$$

Given $g \in L^\infty(\mathbb{D})$, denote by T_g the Toeplitz operator

$$T_g : \mathcal{A}^2(\mathbb{D}) \rightarrow \mathcal{A}^2(\mathbb{D}), \quad T_g f := P_{\mathbb{D}}(gf).$$

Radial Toeplitz operators on the Bergman space

Given $a \in L^\infty([0, 1])$, denote by \tilde{a} its extension onto \mathbb{D} :

$$g(z) := \tilde{a}(z) := a(|z|) \quad (z \in \mathbb{D}).$$

It is easy to see that Toeplitz operators generated by radial symbols are diagonal in the monomial basis:

Diagonalization of radial Toeplitz operators on the Bergman space:

$$T_g e_n = \lambda_a(n) e_n,$$

where

$$\lambda_a(n) = (n + 1) \int_0^1 a(\sqrt{r}) r^n dr.$$

Korenblum, Zhu (1995)

<http://www.mathjournals.org/jot/1995-033-002/1995-033-002-010.html>

From radial Toeplitz operators to eigenvalues' sequences

$$\begin{array}{ccc} T_{\tilde{a}} & \longleftrightarrow & \lambda_a \\ \{T_{\tilde{a}}: a \in L^\infty([0,1])\} & \longleftrightarrow & \Lambda := \{\lambda_a: a \in L^\infty([0,1])\} \\ C^* \text{-alg}\{T_{\tilde{a}}: a \in L^\infty(0,1)\} & \longleftrightarrow & C^* \text{-alg}(\Lambda) \end{array}$$

Natural questions:

What is the closure of Λ ?

What is the C^* -algebra generated by Λ ?

From radial Toeplitz operators to eigenvalues' sequences

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Natural questions:

What is the closure of Λ ?

What is the C^* -algebra generated by Λ ?

These two questions have the same answer: LO.

Key idea: spectral sequences as convolutions

$$\lambda_a(n) = (n+1) \int_0^1 a(\sqrt{r}) r^n dr.$$

Change of variables:

$$n+1 = \exp(x), \quad r = \exp(-\exp(-y)).$$

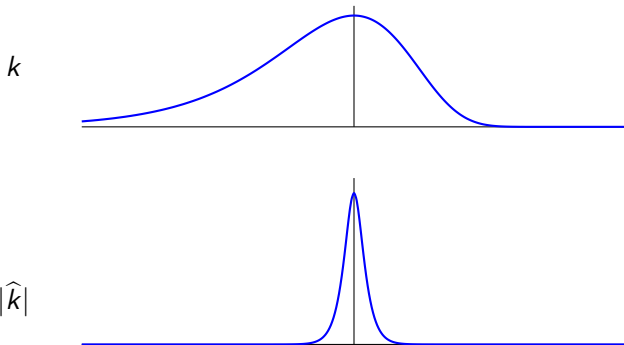
$$\lambda_a(n) = \int_{\mathbb{R}} \underbrace{a(\exp(-\exp(-y)/2))}_{b(y)} k(x-y) dy = (k * b)(x),$$

where

$$k(x) = \frac{e^x}{e^{e^x}}.$$

The convolution kernel satisfies Wiener's condition

$$k(x) = \frac{e^x}{e^{e^x}}, \quad \widehat{k}(t) = \Gamma(1 - 2i\pi t) \neq 0 \quad (t \in \mathbb{R}).$$



Main result for the radial Bergman case

Theorem. $\underbrace{\{\lambda_a: a \in L^\infty([0, 1])\}}_{\Lambda}$ is a dense subset of LO.

As a consequence, $C^*(\Lambda) = \text{LO}$, and the C^* -algebra generated by radial Toeplitz operators on $\mathcal{A}^2(\mathbb{D})$ is isometrically isomorphic to LO.

Suárez (2008) doi:10.1112/blms/bdn042

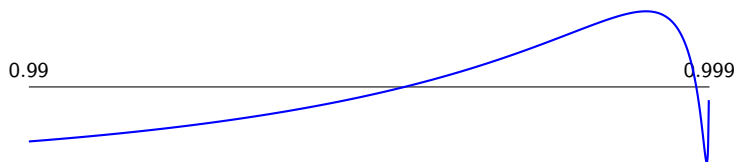
Grudsky, M, Vasilevski (2013) euclid.cma/1356039033

Bauer, Herrera Yañez, Vasilevski (2014) doi:10.1007/s00020-013-2101-1

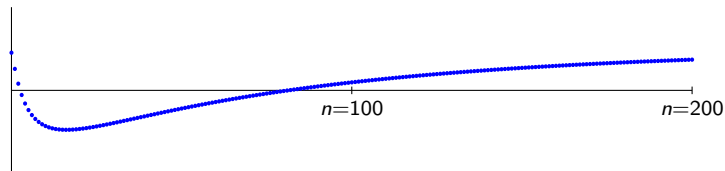
Herrera Yañez, M, Vasilevski (2015) doi:10.1007/s00020-014-2213-2

Example of spectral sequence in radial Bergman case

The function $a(r) = \cos(\log(-2 \log(r)))$ oscillates rapidly near $r = 1$.

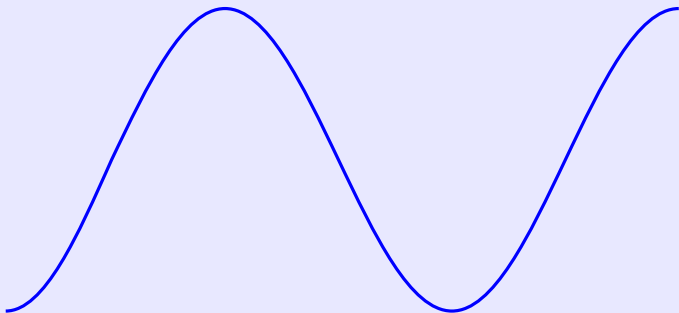


Then $\lambda_a(n) = \operatorname{Re}(\Gamma(1 - i) \exp(i \log(n + 1)))$.



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Approximation

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Bargmann–Segal–Fock space

Gaussian measure $d\gamma(z) = \frac{1}{\pi} e^{-|z|^2} d\mu(z)$.

$\mathcal{A}^2(\mathbb{C}, d\gamma) := \{f \in H(\mathbb{C}) : f \in L^2(\mathbb{C}, d\gamma)\}$.

$P_{\mathbb{C}} :=$ orthogonal projection $L^2(\mathbb{C}, d\gamma) \rightarrow \mathcal{A}^2(\mathbb{C}, d\gamma)$.

Orthonormal basis in $\mathcal{A}^2(\mathbb{C}, d\gamma)$:

$$f_n(z) = \frac{z^n}{\sqrt{n!}}.$$

Given $g \in L^\infty(\mathbb{C})$, denote by T_g the Toeplitz operator

$$T_g : \mathcal{A}^2(\mathbb{C}, d\gamma) \rightarrow \mathcal{A}^2(\mathbb{C}, d\gamma), \quad T_g f := P_{\mathbb{C}}(gf).$$

Radial Toeplitz operators on the Fock space

Given $a \in L^\infty(\mathbb{R}_+)$, denote by \tilde{a} its extension onto \mathbb{C} :

$$g(z) := \tilde{a}(z) := a(|z|) \quad (z \in \mathbb{C}).$$

Diagonalization of radial Toeplitz operators on the Fock space:

$$T_g f_n = \xi_a(n) f_n,$$

where

$$\xi_a(n) = \frac{1}{\sqrt{n!}} \int_{\mathbb{R}_+} a(\sqrt{r}) e^{-r} r^n dr.$$

Grudsky, Vasilevski (2002) doi:10.1007/BF01197858

Key idea: approximate spectral sequences by convolutions

We proved that

$$\xi_a(n) - \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}_+} e^{-2(\sqrt{n}-y)^2} a(y) dy \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let a_{ext} be the function a extended from \mathbb{R}_+ onto \mathbb{R} by zero. Then

$$\xi_a(n) \approx (k * a_{\text{ext}})(\sqrt{n}) \quad \text{as } n \rightarrow \infty,$$

where k is the following Gaussian density:

$$k(x) = \sqrt{\frac{2}{\pi}} e^{-2x^2}.$$

The Fourier transform of k does not vanish:

$$\widehat{k}(t) = e^{-\frac{1}{2}\pi^2 t^2}.$$

Main result for the radial Fock case

Theorem. $\{\xi_a: a \in L^\infty(\mathbb{R}_+)\}$ is a dense subset of RO.

As a consequence, the C^* -algebra generated by radial Toeplitz operators on $\mathcal{A}^2(\mathbb{C}, d\gamma)$ is isometrically isomorphic to RO.

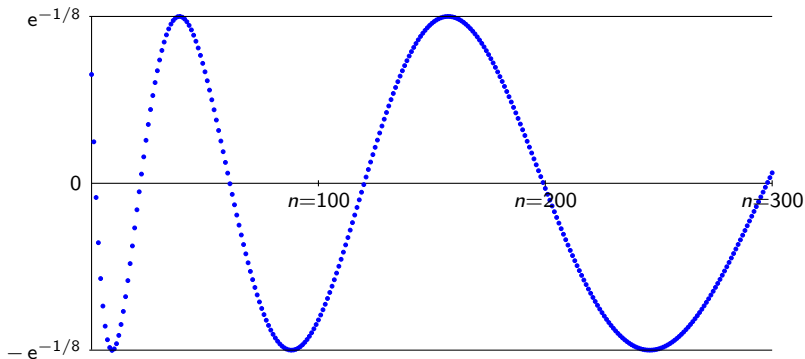
Esmeral, Maximenko (2016) doi:10.1007/s11785-016-0557-0

Example of spectral sequence in radial Fock case

Let $a(r) = \cos(r)$, $r \in \mathbb{R}_+$.

Then the eigenvalue sequence ξ_a has a typical $\sqrt{\cdot}$ -oscillation:

$$\xi_a(n) = {}_1F_1(1+n; 1/2; 1/4) \approx e^{-1/8} \cos(\sqrt{n}) + o(1).$$



Summary

