Avram–Parter and Szegő limit theorems: from weak convergence to uniform approximation

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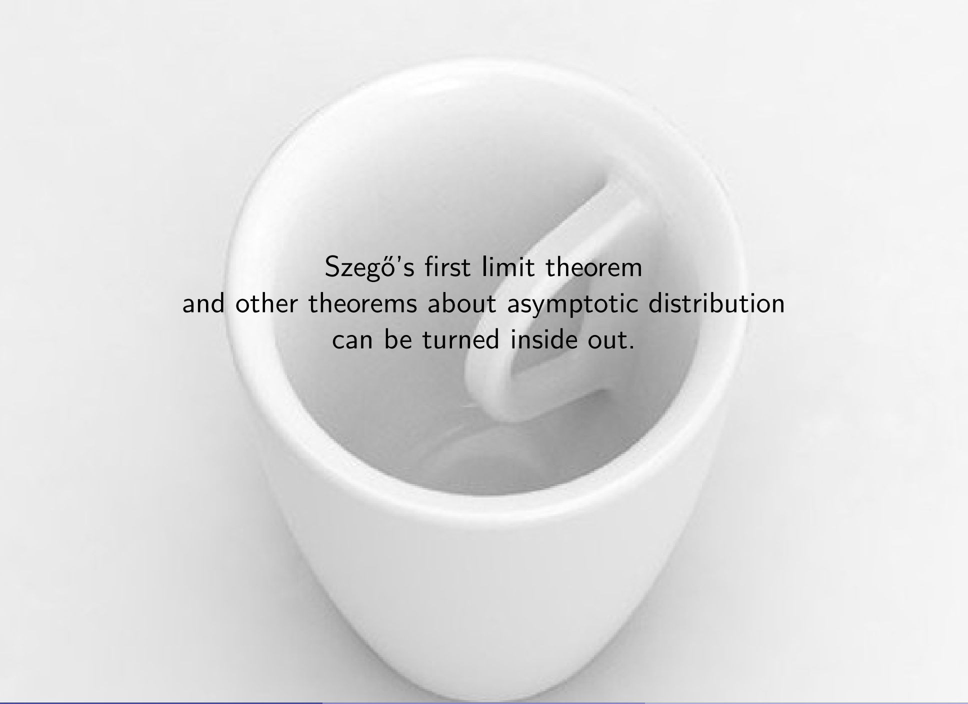
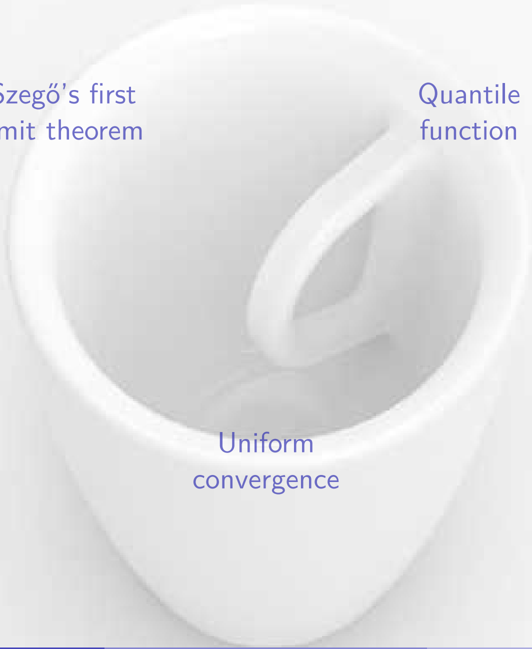


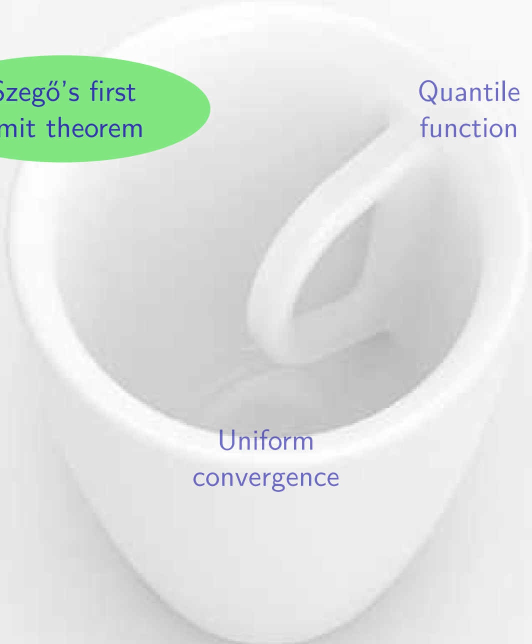
Szegő's first limit theorem
and other theorems about asymptotic distribution
can be turned inside out.



Szegő's first limit theorem

Quantile function

Uniform convergence



Szegő's first limit theorem

Quantile function

Uniform convergence

Toeplitz matrices

$$T_5(a) = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ a_3 & a_2 & a_1 & a_0 & a_{-1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}.$$

It is convenient to think that a_k are the Fourier coefficients of a certain function a defined on $[0, 2\pi]$:

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} a(\theta)\, e^{-ki\theta}\, d\theta.$$

The function a is called the *generating symbol* of the matrices

$$T_n(a) = \left[a_{j-k}\right]_{j,k=1}^{n}.$$

Hermitian Toeplitz matrices, real bounded symbols

We suppose that the generating symbol is bounded and real:

$$a \in L^\infty([0, 2\pi], \mathbb{R}).$$

The corresponding Toeplitz matrices are Hermitian: $\quad a_{-k} = \overline{a_k}, \quad a_0 \in \mathbb{R}.$

$$T_5(a) = \begin{bmatrix} a_0 & \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} \\ a_1 & a_0 & \overline{a_1} & \overline{a_2} & \overline{a_3} \\ a_2 & a_1 & a_0 & \overline{a_1} & \overline{a_2} \\ a_3 & a_2 & a_1 & a_0 & \overline{a_1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}.$$

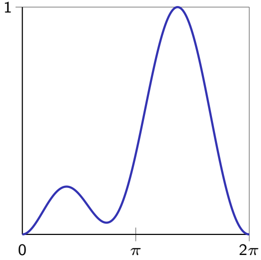
The spectra of $T_n(a)$ "asymptotically fill" $[\operatorname{ess\,inf}(a), \operatorname{ess\,sup}(a)]$:

$$\operatorname{ess\,inf}(a) \le \lambda_1^{(n)} \le \ldots \le \lambda_n^{(n)} \le \operatorname{ess\,sup}(a),$$

$$\operatorname{sp}(T_n(a)) \to [\operatorname{ess\,inf}(a), \operatorname{ess\,sup}(a)] \quad \text{(in Hausdorff distance).}$$

Behavior of the eigenvalues of Hermitian Toeplitz matrices

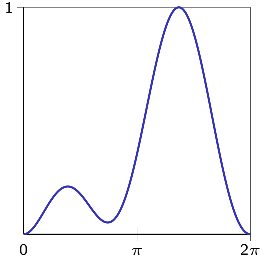
Graph of a



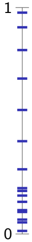
Eigenvalues of $T_8(a)$

Behavior of the eigenvalues of Hermitian Toeplitz matrices

Graph of a

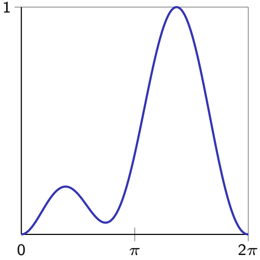


Eigenvalues of $T_{16}(a)$

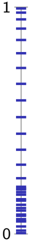


Behavior of the eigenvalues of Hermitian Toeplitz matrices

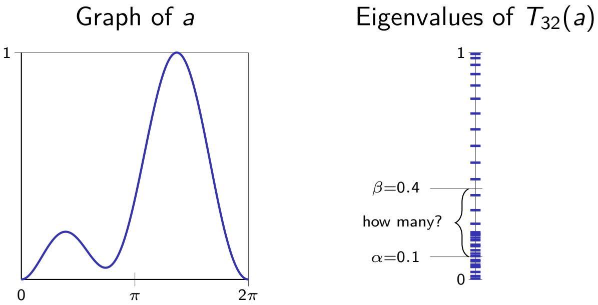
Graph of a



Eigenvalues of $T_{32}(a)$



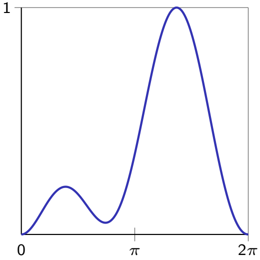
Behavior of the eigenvalues of Hermitian Toeplitz matrices



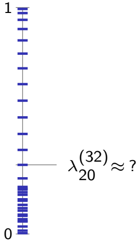
First question: How many eigenvalues are in $[\alpha, \beta]$?

Behavior of the eigenvalues of Hermitian Toeplitz matrices

Graph of a



Eigenvalues of $T_{32}(a)$



First question: How many eigenvalues are in $[\alpha, \beta]$?

Second question: $\lambda_j^{(n)} \approx ?$

Szegő's first limit theorem (1920)

generating symbol
$a \in L^\infty([0, 2\pi], \mathbb{R})$

test function
$\varphi \in C(\mathbb{R})$

$$\frac{1}{n}\sum_{j=1}^{n}\varphi(\lambda_j^{(n)}) \longrightarrow \frac{1}{2\pi}\int_0^{2\pi}\varphi(a(\theta))\,\mathrm{d}\theta$$

Another form of the Szegő's first limit theorem

generating symbol
$a \in L^\infty([0, 2\pi], \mathbb{R})$

$\alpha \; < \; \beta$
$a(\theta) \neq \alpha, \beta$ a.e.

$$\frac{\#\{j\colon\ \alpha \le \lambda_j^{(n)} \le \beta\}}{n} \quad \longrightarrow \quad \frac{\mu_{\mathbb{R}}(a^{-1}([\alpha, \beta]))}{2\pi}$$

Another form of the Szegő's first limit theorem

distribution of the eigenvalues of Hermitian Toeplitz matrices

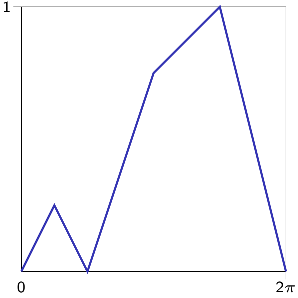
generating symbol
$a \in L^\infty([0, 2\pi], \mathbb{R})$

$v \in \mathbb{R}$
$a(\theta) \neq v$ a.e.

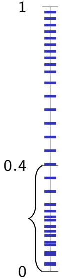
$$\frac{\#\{j\colon\ \lambda_j^{(n)} \le v\}}{n} \longrightarrow \frac{\mu_{\mathbb{R}}(\{\theta \in [0, 2\pi]\colon\ a(\theta) \le v\})}{2\pi}$$

Example to illustrate Szegő's first limit theorem

Graph of a



Eigenvalues of $T_{32}(a)$



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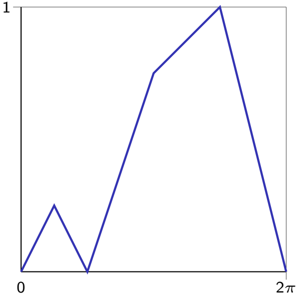

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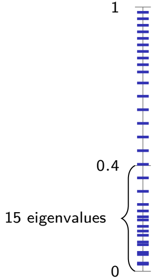

$$\frac{15}{32} \approx 0.469$$

Example to illustrate Szegő's first limit theorem

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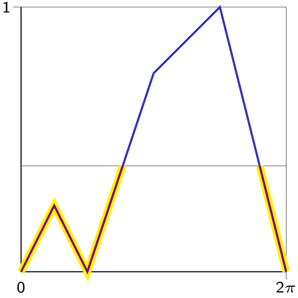
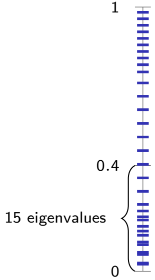


Eigenvalues of $T_{32}(a)$



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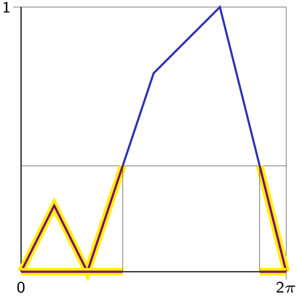
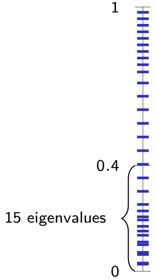


Eigenvalues of $T_{32}(a)$



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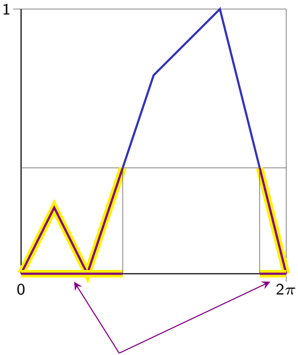

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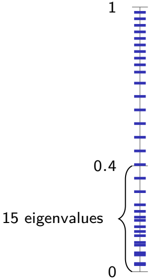


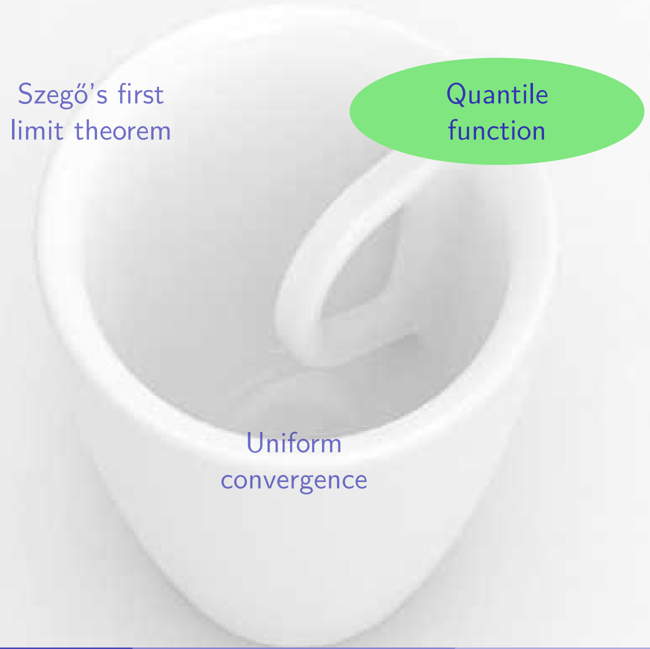
$$\frac{\mu_{\mathbb{R}}\{\theta:\ \ a(\theta)\le 0.4\}}{2\pi} = 0.483$$

$$\frac{15}{32} \approx 0.469$$

Szegő found an approximate answer to the first question:
how many eigenvalues belong to a given interval?

The second question was still open:

$$\lambda_j^{(n)} \approx ?$$


Szegő's first limit theorem
Quantile function
Uniform convergence

Definition of the quantile function

$\mathcal{BPM}(\mathbb{R}) :=$ Borel probability measures over $\mathbb{R}$.

Given $\mu \in \mathcal{BPM}(\mathbb{R})$, one defines:
the cumulative distribution function $F_\mu \colon \mathbb{R} \to [0, 1]$,

$$F_\mu(v) := \mu(-\infty, v],$$

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the quantile function $Q_\mu \colon (0,1) \to \mathbb{R}$,

$$Q_\mu(p) := \inf\{v \in \mathbb{R} \colon \; F_\mu(v) \geq p\},$$

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$$Q_\mu(p) := \inf\{v \in \mathbb{R}\colon\ \ F_\mu(v) \ge p\},$$

and the support of μ:

$$\operatorname{supp}(\mu) := \{v \in \mathbb{R}\colon \quad \forall \varepsilon > 0 \quad \mu(v - \varepsilon, v + \varepsilon) > 0\}.$$

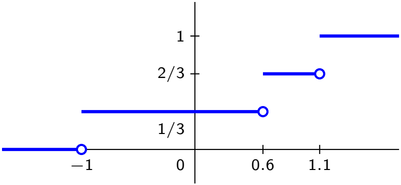
Quantile function associated to a list of real numbers

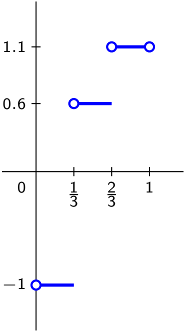
$$X = (-1,\ 0.6,\ 1.1).$$

Associate the weight $1/3$ to each one of the elements of X:

$$\mu(\{-1\}) = \mu(\{0.6\}) = \mu(\{1.1\}) = \frac{1}{3}.$$

The corresponding cdf and the quantile function:



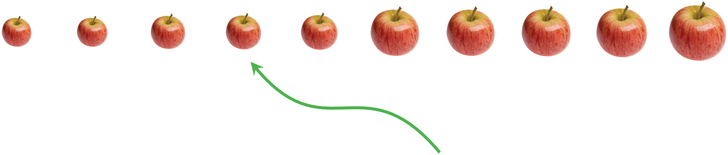


Quantile function associated to a list of real numbers

118 100 195 166 164 123 102 172 164 117

The same numbers in the ascending order ($\alpha_1 \le \alpha_2 \le \ldots \le \alpha_{10}$):

100 102 117 118 123 164 164 166 172 195



$$Q_\alpha(1/3) = \alpha_{\lceil 10/3 \rceil} = \alpha_4 = 118.$$

$$Q_\alpha(p) = \alpha_{\lceil np \rceil}.$$

Quantile function associated to a function

Let $a \in L^\infty([0, 2\pi], \mathbb{R})$.

Pushforward measure $\mu \in \mathcal{BPM}(\mathbb{R})$:

$$\mu(B) := \frac{1}{2\pi} \mu_{\mathbb{R}}(a^{-1}(B)).$$

$F_a :=$ the cumulative distribution function of a:

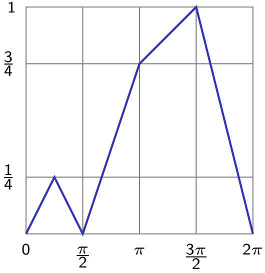
$$F_a(v) := \frac{1}{2\pi} \mu_{\mathbb{R}} \{\theta \in [0, 2\pi] \colon \ a(\theta) \le v\}, \qquad v \in \mathbb{R}.$$

$Q_a :=$ the corresponding quantile function:

$$Q_a(p) := \inf\{v \in \mathbb{R} \colon \ F_a(v) \ge p\}, \qquad p \in (0, 1).$$

Construction of the quantile function associated to a piecewise-linear real function

Graph of a



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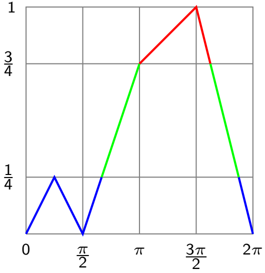

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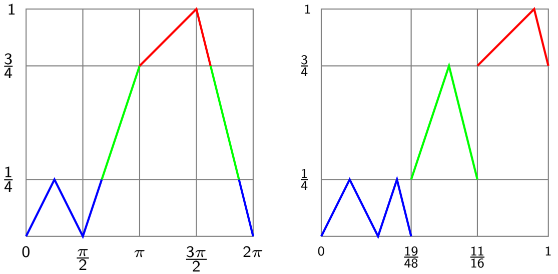

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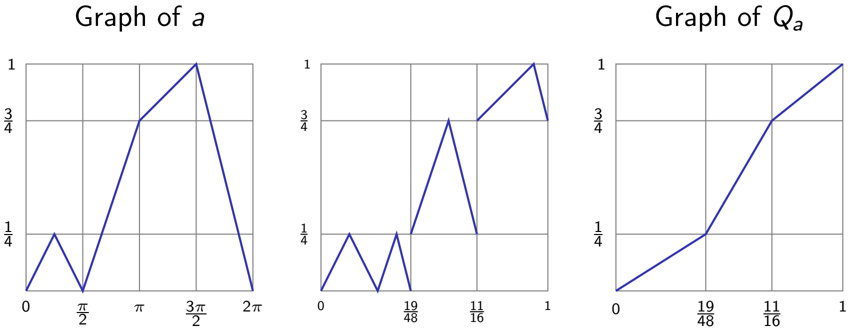

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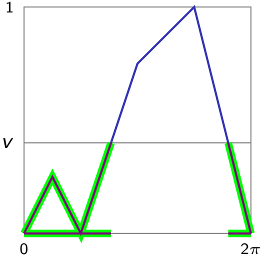
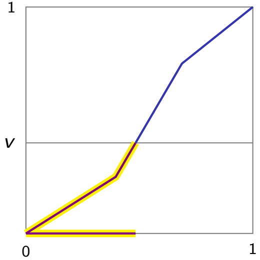


Graph of Q_a

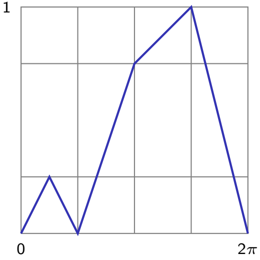


a and Q_a are identically distributed:

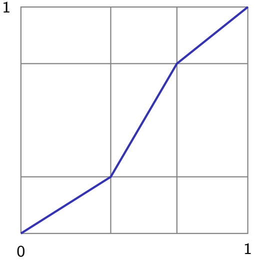
$$\frac{1}{2\pi}\mu_{\mathbb{R}}\{\theta \in [0,2\pi]\colon\ a(\theta) \le v\} = \mu_{\mathbb{R}}\{p \in [0,1]\colon\ Q_a(p) \le v\}$$

Construction of the quantile function associated to a piecewise-linear real function

Graph of a



Graph of Q_a



$$a \xrightarrow{\text{reordering in Lebesgue-style}} Q_a$$

Criterion for continuity of the quantile function

Let $\mu \in \mathcal{BPM}(\mathbb{R})$ with compact supp(μ).
Then the following conditions are equivalent:

supp(μ) is connected

$\updownarrow$

F_μ is strictly increasing

$\updownarrow$

Q_μ is continuous

Convergence in distribution ($\mu_n \rightsquigarrow \Lambda$)

(convergence in law, weak convergence)

Let $\Lambda \in \mathcal{BPM}(\mathbb{R})$ and let $(\mu_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{BPM}(\mathbb{R})$.
Then the following conditions are equivalent.

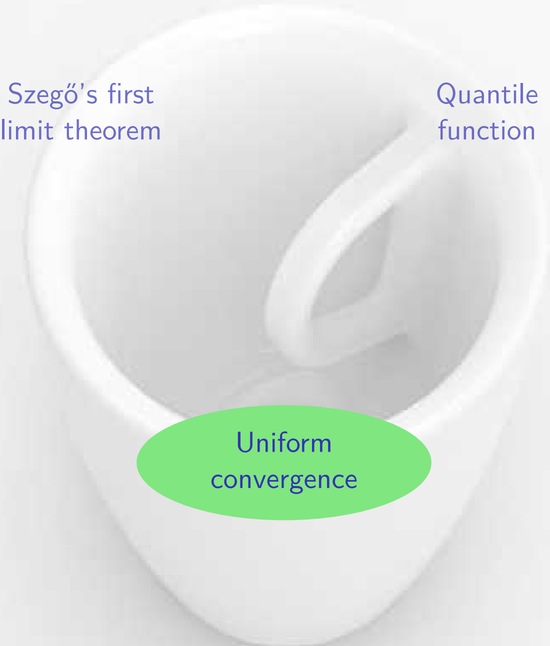
$$\forall \varphi \in C_b(\mathbb{R}) \qquad \lim_{n\to\infty} \int_{\mathbb{R}} \varphi \, d\mu_n = \int_{\mathbb{R}} \varphi \, d\Lambda$$

$$\Updownarrow$$

$$\forall v \in \mathcal{C}(F_\Lambda) \qquad \lim_{n\to\infty} F_{\mu_n}(v) = F_\Lambda(v)$$

$$\Updownarrow$$

$$\forall p \in \mathcal{C}(Q_\Lambda) \qquad \lim_{n\to\infty} Q_{\mu_n}(p) = Q_\Lambda(p)$$


Szegő's first
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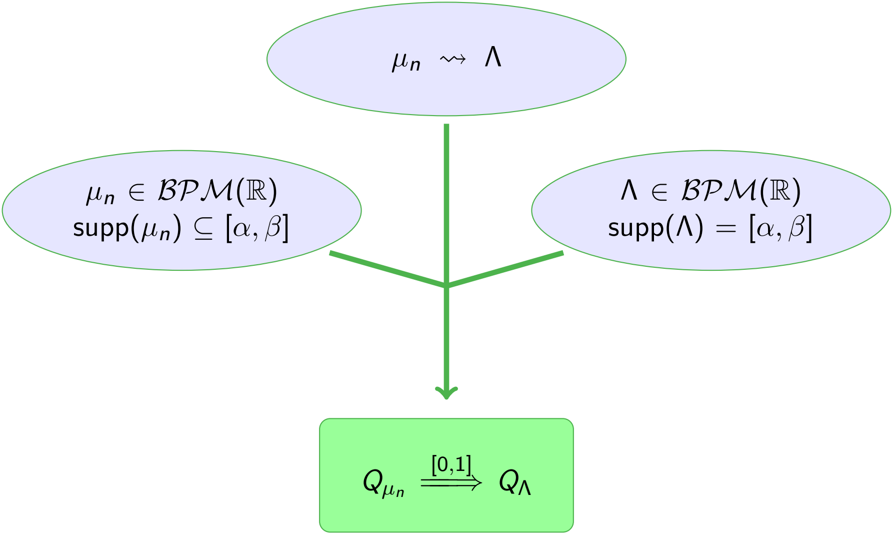
Main result


$\mu_n \rightsquigarrow \Lambda$
$\mu_n \in \mathcal{BPM}(\mathbb{R})$
$\operatorname{supp}(\mu_n) \subseteq [\alpha, \beta]$
$\Lambda \in \mathcal{BPM}(\mathbb{R})$
$\operatorname{supp}(\Lambda) = [\alpha, \beta]$
$Q_{\mu_n} \overset{[0,1]}{\Longrightarrow} Q_\Lambda$

Application to Toeplitz matrices: uniform approximation of the eigenvalues

$a \in L^\infty([0, 2\pi], \mathbb{R})$

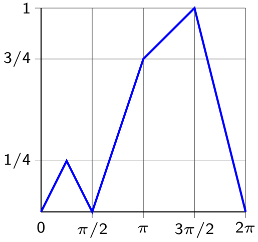
$\mathcal{R}(a) = [\alpha, \beta]$

$$\max_{1 \le j \le n} \left| \lambda_j^{(n)} - Q_a(j/n) \right| \longrightarrow 0$$

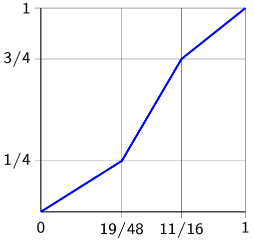
First example

continuous piecewise-linear generating symbol

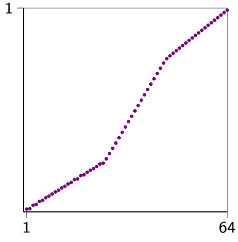
Graph of a



Graph of Q_a



Eigenvalues of $T_{64}(a)$



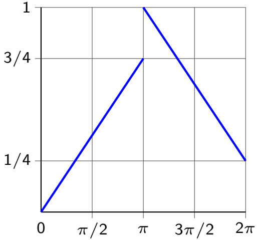
Every eigenvalue $\lambda_j^{(n)}$ is shown as a point $\left(\frac{j}{n},\ \lambda_j^{(n)}\right)$.

The third picture mimics the second one.

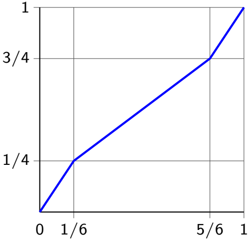
Second example

a is not continuous, but $\mathcal{R}(a)$ is connected

Graph of a



Graph of Q_a



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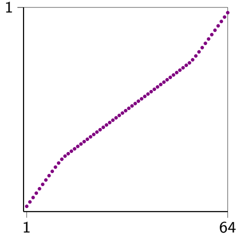
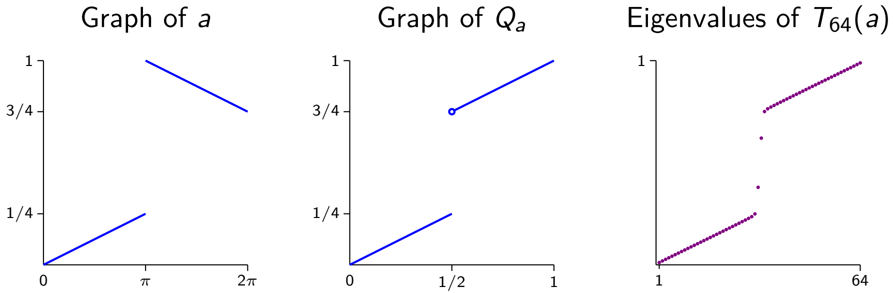


In this example, $\lambda_j^{(n)}$ is also uniformly approximated by $Q_a(j/n)$ as $n \to \infty$.

Third example

If $\mathcal{R}(a)$ is not connected, then the uniform convergence fails



In this example, $\lambda^{(n)}_{\lfloor n/2 \rfloor}$ can not be approximated by values of Q_a.

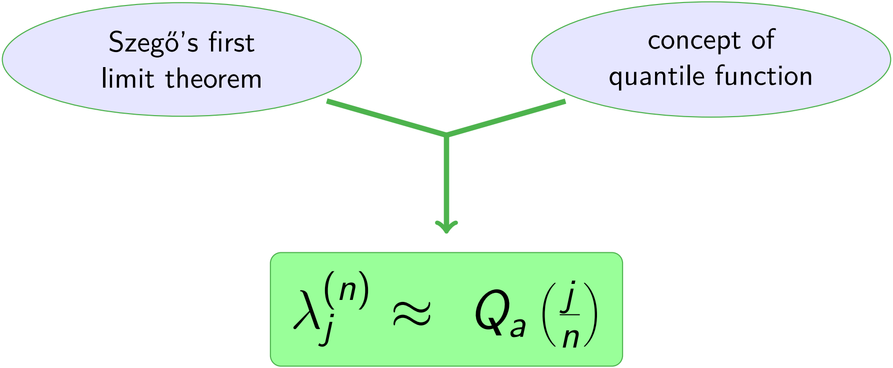
Summary

Szegő's first limit theorem

concept of quantile function

$$\lambda_j^{(n)} \approx Q_a\left(\frac{j}{n}\right)$$

Summary



Similar results, without using the terminology of quantile function:

Di Benedetto, Serra, Fiorentino (1993): pointwise convergence.

Trench (2012): convergence in L^1 sence.

More applications

There are other results about asymptotic distribution:

- Avram–Parter theorem,
- Szegő type theorems for locally Toeplitz matrices,
- Lévy's arcsine law for random walks,
- Weyl's theorem about uniformly distributed sequences.

Applying the concept of quantile function
one easily deduces corollaries about uniform approximation.

Uniform approximation of the singular values (quantile version of Avram–Parter theorem)

$a \in L^\infty([0, 2\pi], \mathbb{C})$

$\mathcal{R}(|a|) = [0, \|a\|_\infty]$

$$\max_{1 \le j \le n} \left| s_j^{(n)} - Q_{|a|}(j/n) \right| \longrightarrow 0$$

Interactive visualization

http://www.egormaximenko.com

Thanks for attention!