Analysis of translation-invariant operators via the Fourier transform of the reproducing kernel

Egor Maximenko, joint results with
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Outline

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A general problem (we cannot solve it)

Let

X be a set,

H be a reproducing kernel Hilbert space over X,

G be a locally compact group,

$\tau\colon G \to \operatorname{Sym}(X)$ be a group action,

$(\rho(g))_{g\in G}, \quad \rho(g)\, f := f \circ \tau(g^{-1})$ be a unitary representation of G in H.

Problem: describe the W*-algebra defined as the centralizer of ρ,

$$\mathcal{C}(\rho) \;:=\; \Big\{ S \in \mathcal{B}(H)\colon \quad \forall g \in G \quad S\,\rho(g) = \rho(g)\,S \Big\}.$$

A general idea

Apply the Fourier transform to the reproducing kernel along the orbits of the group action:

$$\int_G K_z(\tau(g)(w))\,\psi(g)^*\,\mathrm{d}\nu_G(g), \qquad \psi \in \text{irreducible representations of } G.$$

We hope that the obtained operator-valued function is useful to describe $\mathcal{C}(\rho)$.

Our scheme for type-type domains $G \times Y$

Crispin Herrera-Yañez, Egor A. Maximenko, Gerardo Ramos-Vazquez (2022):
Translation-invariant operators in reproducing kernel Hilbert spaces.
Integral Equ. Oper. Theory. DOI: 10.1007/s00020-022-02705-4.

Our paper is inspired by various works of Vasilevski and other mathematicians.

Nikolai L. Vasilevski (1999):
On Bergman-Toeplitz operators with commutative symbol algebras.
Integral Equ. Oper. Theory. DOI: 10.1007/BF01332495.

Our assumptions

- $X = G \times Y$,
- G is an abelian locally compact group, metrizable, and σ-compact,
- ν is a Haar measure on G,
- (Y, λ) is a σ-finite measure space,
- $L^2(G \times Y)$ is separable,
- $H \leq L^2(G \times Y)$,
- H is an RKHS; we denote the RK by $(K_{x,y})_{x\in G,\ y\in Y}$,

Our assumptions

- G acts in $G \times Y$ by
$$\tau_{G\times Y}(g)\colon \quad (x,y) \mapsto (g+x,y),$$
- $\rho_{G\times Y}$ acts in $L^2(G \times Y)$
$$(\rho_{G\times Y}(a)f)(x,y) := f(x-a,y),$$
- H is invariant under $\rho_{G\times Y}$,
- $\forall y \in Y \quad \sup_{v\in Y} \int_G |K_{(0,y)}(u,v)|\,\mathrm{d}\nu(u) < +\infty.$

Criterion that H is shift-invariant

$P :=$ the orthogonal projection in $L^2(G \times Y)$ whose image is H.

Proposition

The following conditions are equivalent.

(a) $\rho_{G\times Y}(a)(H) \subseteq H$ for every a in G.

(b) $P\rho_{G\times Y}(a) = \rho_{G\times Y}(a)P$ for every a in G.

(c) $K_{x,y}(u,v) = K_{0,y}(u-x,v)$ for every x, y in G and every y, v in Y.

(d) $\rho_{G\times Y}(a)K_{x,y} = K_{a+x,y}$ for every a, x in G and every y in Y.

Let $\rho_H(a)\colon H \to H$ be the compression of $\rho_{G\times Y}(a)$.

Decomposition of H

$$\widehat{P} := (F \otimes I)P(F \otimes I)^*, \qquad \widehat{H} := (F \otimes I)(H).$$

$$\widehat{P} = \int_{\widehat{G}}^{\oplus} \widehat{P}_\xi \, \mathrm{d}\widehat{\nu}(\xi).$$

For each ξ in $\widehat{G}$,

$$\widehat{H}_\xi := \widehat{P}_\xi(L^2(Y)).$$

$$\Omega := \{\xi \in \widehat{G} \colon \ \dim(\widehat{H}_\xi) > 0\}.$$

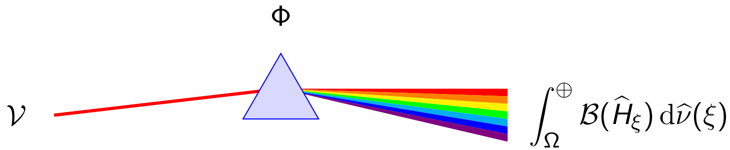
$$\widehat{H} = \int_{\Omega}^{\oplus} \widehat{H}_\xi \, \mathrm{d}\widehat{\nu}(\xi).$$

Decomposition of $\mathcal{V} := \mathcal{C}(\rho_H)$

Let $\Phi\colon H \to \widehat{H}$ be the compression of $F \otimes I$.

Theorem

$$\Phi\, \mathcal{V}\, \Phi^* = \int_{\Omega}^{\oplus} \mathcal{B}(\widehat{H}_\xi)\, \mathrm{d}\widehat{\nu}(\xi).$$



Constructive description of the fibers $\widehat{H}_\xi$

$$L_{\cdot,y} := (F \otimes I)K_{0,y}, \qquad \text{i.e.,} \qquad L_{\xi,y}(v) := \int_G K_{(0,y)}(u,v)\,\overline{\xi(u)}\,\mathrm{d}\nu(u).$$

Theorem

For every ξ in Ω, the family $(L_{\xi,y})_{y\in Y}$ is the reproducing kernel of $\widehat{H}_\xi$.

Idea of the proof: convolution theorem + Fubini + Moore–Aronszajn theorem.

Constructive criterion for the commutativity of $\mathcal{V}$

Theorem

The following conditions are equivalent.

(a) $\mathcal{V}$ is commutative.

(b) $d_\xi := \dim(\widehat{H}_\xi) = 1$ for every ξ in Ω.

(c) $|L_{\xi,y}(v)|^2 = L_{\xi,y}(y) L_{\xi,v}(v)$ for every ξ in Ω and every y, v in Y.

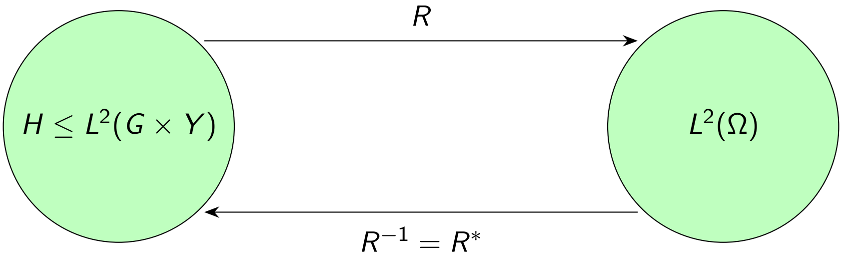
(d) There exists a family $(q_\xi)_{\xi\in\Omega}$ in $L^2(Y)$ such that the function $(\xi, v) \mapsto q_\xi(v)$ is measurable, $\widehat{H}_\xi = \mathbb{C} q_\xi$, $\|q_\xi\| = 1$, and

$$L_{\xi,y}(v) = \overline{q_\xi(y)} q_\xi(v) \qquad (\xi \in \Omega,\ y, v \in Y).$$

Isometric isomorphism $R\colon H \to L^2(\Omega)$ in the commutative case

Suppose that $\dim(\widehat{H}_\xi) = 1$, i.e.,
we have a family $(q_\xi)_{\xi\in\Omega}$ such that $\widehat{H}_\xi = \mathbb{C}q_\xi$ y $\|q_\xi\| = 1$.

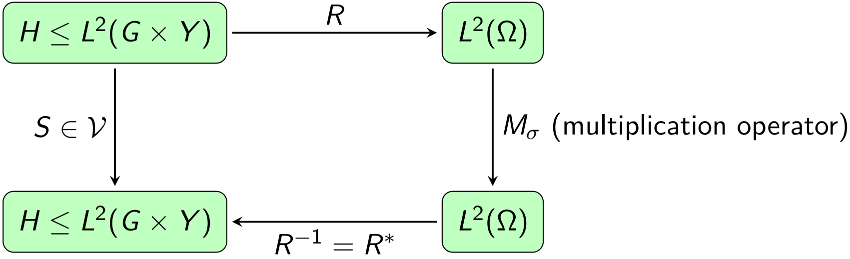
$$(Rf)(\xi) := \langle (\Phi f)(\xi, \cdot), q_\xi \rangle_{L^2(Y)}.$$



Diagonalization of translation-invariant operators in the case $d_\xi = 1$

Proposition

Suppose that $\dim(\widehat{H}_\xi) = 1$ for every ξ in Ω. Entonces $\mathcal{V} \cong L^\infty(\Omega)$.



The case of finite-dimensional fibers

Suppose that

$$\forall \xi \in \Omega \qquad d_\xi := \dim(\widehat{H}_\xi) < +\infty.$$

Let $(q_{j,\xi})_{j\in\mathbb{N},\xi\in\Omega}$ be a measurable basis family for the spaces $\widehat{H}_\xi$.

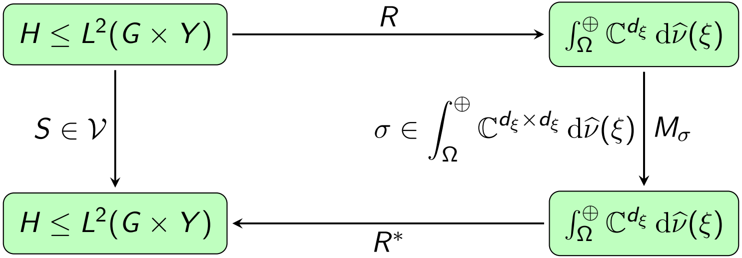
$$L_{\xi,y}(v) = \sum_{j=1}^{d_\xi} \overline{q_{j,\xi}(y)}\, q_{j,\xi}(v).$$

Then

$$\Phi H = \widehat{H} = \int_\Omega^\oplus \widehat{H}_\xi \,\mathrm{d}\widehat{\nu}(\xi) \cong \int_\Omega^\oplus \mathbb{C}^{d_\xi} \,\mathrm{d}\widehat{\nu}(\xi).$$

From translation-invariant operators to matrix families

$$R\colon H \to \int_\Omega^\oplus \mathbb{C}^{d_\xi}\,\mathrm{d}\widehat{\nu}(\xi), \qquad (Rf)(\xi) := \Big[\langle (\Phi f)(\xi,\cdot),\ q_{j,\xi}\rangle_{L^2(Y)}\Big]_{j=1}^{d_\xi}.$$



Matrix families corresponding to Toeplitz operators with translation-invariant generating symbols

Corollary

Let $\psi \in L^\infty(Y)$,

$$\varphi(x, y) = \psi(y).$$

Then $T_\varphi \in \mathcal{V}$, $RT_\varphi R^* = M_{\gamma_\psi}$,

$$\gamma_\psi(\xi) := \left[\int_Y \psi(v)\overline{q_{j,\xi}(v)}q_{k,\xi}(v)\,\mathrm{d}\lambda(v)\right]_{j,k=1}^{d_\xi}.$$

Outline

Separately radial case on the Bergman space

Raul Quiroga-Barranco, Nikolai Vasilevski (2007):
Commutative C*-algebras of Toeplitz operators on the unit ball, I.
Bargmann-type transforms and spectral representations of Toeplitz operators.
Integral Equ. Oper. Theory. DOI: 10.1007/s00020-007-1537-6.

They worked with maximal abelian subgroups of the Möbius group
and diagonalized Toeplitz operators with group-invariant symbols.

Jointly with Alejandro Hernández Arteaga,
we studied three of these cases using the scheme above.

In this talk, we will see the separately radial (= quasi-radial) case.

$\mathcal{A}^2(\mathbb{B}_n, \mu_{n,\alpha})$, the analytic Bergman space

$$\mathbb{B}_n := \{z \in \mathbb{C}^n \colon \; |z| < 1\}.$$

$$\mathrm{d}\mu_{n,\alpha}(z) = c_{n,\alpha}\,(1 - |z|^2)^{\alpha}\,\mathrm{d}\mu_{2n}(z), \qquad c_{n,\alpha} = \frac{\Gamma(n+\alpha+1)}{\pi^n\,\Gamma(\alpha+1)}.$$

$\mathcal{A}^2 = \mathcal{A}^2(\mathbb{B}_n, \mu_{n,\alpha}) :=$ holomorphic functions belonging to $L^2(\mathbb{B}_n, \mu_{n,\alpha})$.

Orthonormal basis: $$b_j(z) = \sqrt{\frac{\Gamma(n+|j|+\alpha+1)}{j!\,\Gamma(n+\alpha+1)}}\,z^j, \qquad j \in \mathbb{N}_0^n.$$

Reproducing kernel of $\mathcal{A}^2$: $$K_z^{\mathcal{A}^2}(w) = \frac{1}{(1 - \langle w, z\rangle)^{n+1+\alpha}}.$$

Group $\mathbb{R}_{2\pi}^n$ and its dual group

$G := \mathbb{R}_{2\pi}^n \cong \mathbb{T}^n$, where $\mathbb{R}_{2\pi} := \mathbb{R}/(2\pi\mathbb{Z})$.

$\nu :=$ the normalized Haar measure on G.

$$\int_G f \, \mathrm{d}\nu = \frac{1}{(2\pi)^n} \int_{[0,2\pi)^n} f(g) \, \mathrm{d}\mu_n(g).$$

$\widehat{G} = \mathbb{Z}^n$ with the counting measure $\widehat{\nu}$.

Pairing between G and $\widehat{G}$:

$$E(u + 2\pi\mathbb{Z}^n, \xi) = \mathrm{e}^{\mathrm{i}\,\langle u,\xi\rangle} \, .$$

Rotations acting in $\mathcal{A}^2$

$G = \mathbb{R}^n_{2\pi} \cong \mathbb{T}^n$.

Action of G on $\mathbb{B}_n$:

$$\tau_{\mathrm{rot}}(g)(z) := \left(e^{i g_1} z_1, \ldots, e^{i g_n} z_n\right).$$

Unitary representation of G in $\mathbb{B}_n$:

$$(\rho_{\mathrm{rot}}(g)f)(z) := f(\tau_{\mathrm{rot}}(-g)z) = f\left(e^{-i g_1} z_1, \ldots, e^{-i g_n} z_n\right).$$

Passing to the polar coordinates

Let Y be the base of $\mathbb{B}_n$ considered as a Reinhard domain:

$$Y = \left\{ y \in [0, +\infty)^n : \quad |y|^2 < 1 \right\}.$$

We consider Y with the Lebesgue measure μ_n.

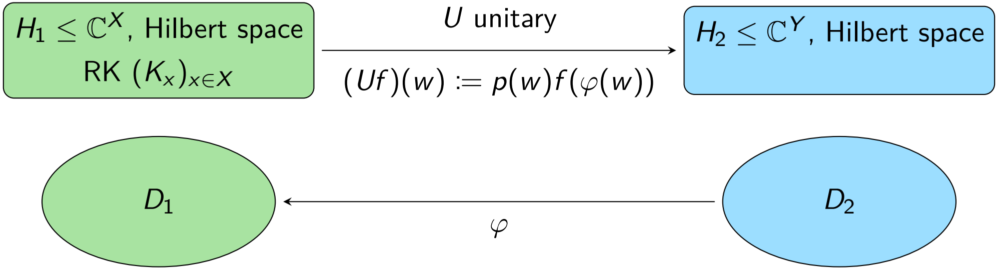
$$\varphi_{\text{polar}} \colon G \times Y \to \mathbb{B}_n, \qquad \varphi_{\text{polar}}(u, v) = \left(v_1 \, \mathrm{e}^{\mathrm{i}\, u_1}, \ldots, v_n \, \mathrm{e}^{\mathrm{i}\, u_n}\right).$$

For every f in $\mathcal{A}^2$, we define $U_{\text{polar}} f \in L^2(G \times Y)$,

$$(U_{\text{polar}} f)(u, v) := \sqrt{(2\pi)^n \, c_{n,\alpha} \, v_1 \cdots v_n} \, \left(1 - |v|^2\right)^{\alpha/2} f(\varphi_{\text{polar}}(u, v)).$$

$H := U_{\text{polar}}(\mathcal{A}^2)$. $\quad U_{\text{polar}} \colon \mathcal{A}^2 \to H$ is an isometric isomophism.

Transformation of RK by a weighted change of variable



Then the following function is the RK of H_2:

$$\widetilde{K}_z(w) = \overline{p(z)}\, p(w)\, K_{\varphi(z)}(\varphi(w)).$$

Passing to the polar coordinates

The reproducing kernel of H is

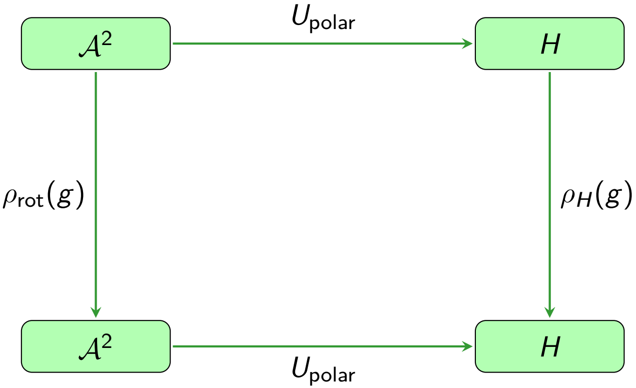
$$K_{x,y}(u,v) = \frac{(2\pi)^n \, c_{n,\alpha} \, (1-|y|^2)^{\alpha/2} \, (1-|v|^2)^{\alpha/2} \, \prod_{k=1}^n \sqrt{y_k v_k}}{\left(1 - \sum_{k=1}^n y_k \, v_k \, e^{\mathrm{i}\,(u_k - x_k)}\right)^{n+\alpha+1}}.$$

We see that $K_{x,y}(u,v) = K_{0,y}(u-x,v)$.

Furthermore, U_{polar} intertwines ρ_{rot} with horizontal translations:

$$\forall g \in G \qquad U_{\text{polar}} \, \rho_{\text{rot}}(g) = \rho_{G\times Y}(g) \, U_{\text{polar}}.$$

U_{polar} intertwines the rotations acting in $\mathcal{A}^2$ with the horizontal translations acting in H



Computation of L

$K_{0,y}(\cdot\,, v)$ decomposes into the Fourier series:

$$K_{0,y}(u,v) = (2\pi)^n\, c_{n,\alpha}\, (1-|v|^2)^{\alpha/2}\, (1-|y|^2)^{\alpha/2} \prod_{k=1}^{n} \sqrt{v_k y_k} \times$$
$$\times \sum_{\xi \in \mathbb{N}_0^n} \frac{\Gamma(n+|\xi|+\alpha+1)}{\xi!\,\Gamma(n+\alpha+1)}\, y^\xi\, v^\xi\, \mathrm{e}^{\mathrm{i}\,\langle u,\xi\rangle}\,.$$

For ξ in $\mathbb{N}_0^n$, the ξth Fourier coefficient is $L_{\xi,y}(v) = \overline{q_\xi(y)}\, q_\xi(v)$, where

$$q_\xi(v) = \sqrt{\frac{2^n\,\Gamma(n+|\xi|+\alpha+1)}{\xi!\,\Gamma(\alpha+1)}}\, (1-|v|^2)^{\alpha/2} \prod_{k=1}^{n} v_k^{\xi_k+\frac{1}{2}}.$$

Conclusions

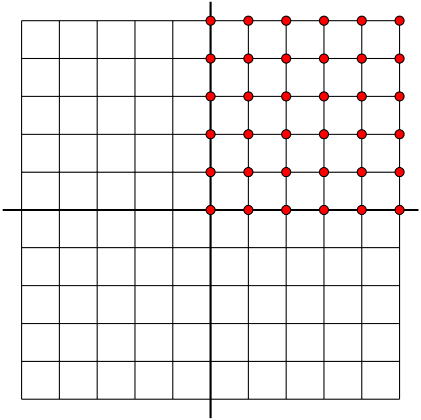
In this example:

$$\Omega = \mathbb{N}_0^n,$$

$$d_\xi = 1 \text{ for } \xi \text{ in } \mathbb{N}_0^n,$$

$$\mathcal{C}(\rho_{\mathrm{rot}}) \cong \mathcal{C}(\rho_H) = \mathcal{V} \cong L^\infty(\mathbb{N}_0^n).$$

$\mathcal{V}$ is commutative.



The eigenvalues of separately radial Toeplitz operators in $\mathcal{A}^2(\mathbb{B}_n)$

We suppose that $\varphi \in L^\infty(Y)$.

$$\gamma_\varphi(\xi) = \int_Y \varphi(v)\,|q_\xi(v)|^2\,\mathrm{d}\mu_n(v)$$

$$= \frac{\Gamma(n + |\xi| + \alpha + 1)}{\xi!\,\Gamma(\alpha + 1)} \int\limits_{|t|_1<1} \varphi(\sqrt{t})\,(1 - |t|_1)^\alpha\, t^\xi\,\mathrm{d}\mu_n(t).$$

Here $|t|_1 = t_1 + \cdots + t_n$.

This formula coincides with the formula found by Quiroga-Barranco and Vasilevski.

Separately radial operators on the pluriharmonic Bergman space

Reproducing kernel: $$\frac{1}{(1-\langle w,z\rangle)^{n+1+\alpha}}+\frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}-1.$$

The analysis of separately radial operators is similar to the analytic case, but

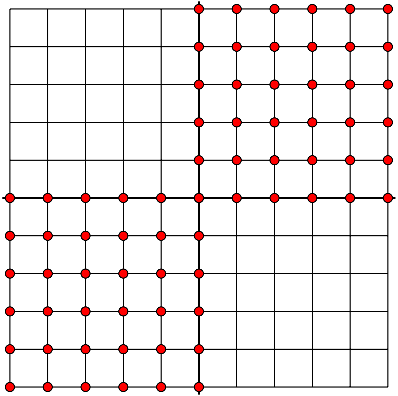
$$\Omega=\mathbb{N}_0^n\cup(-\mathbb{N}_0)^n=\{0,1,2,\dots\}^n\cup\{0,-1,-2,\dots\}^n.$$

$\mathcal{V}\cong L^\infty(\Omega)$. C*-alg(sep. radial Toeplitz operators) is **not** weakly dense in $\mathcal{V}$.

- Jingyu Yang, Liu Liu, Yufeng Lu (2013).
- Maribel Loaiza, Carmen Lozano (2014).

Separately radial operators on the pluriharmonic Bergman space

$$\Omega = \mathbb{N}_0^n \cup (-\mathbb{N}_0)^n.$$



Outline

This is a beginning of a joint work with
Gerardo Ramos Vazquez, Armando Sánchez Nungaray, and Erick Lee Guzmán.

At the moment, we have applied the $G \times Y$ scheme
to the vertical operators on the m-analytic Fock space $\mathcal{F}_m(\mathbb{C}^1)$.
Our results are similar to the following two papers.

- Nikolai L. Vasilevski (2000):
  Poly-Fock spaces.
- Armando Sánchez-Nungaray, Carlos González-Flores, Raquiel Rufino López-Martínez, Jorge Luis Arroyo-Neri (2018):
  Toeplitz operators with horizontal symbols acting on the poly-Fock spaces.

Polyanalytic Bargmann–Segal–Fock space

$\mathcal{F}_m :=$ m-analytic functions $\mathbb{C} \to \mathbb{C}$, square integrable with weight $e^{-|z|^2}$.

$$\|f\|_{\mathcal{F}_m} := \left(\frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 \, e^{-|z|^2} \, \mathrm{d}\mu_2(z) \right)^{1/2},$$

Nour eddine Askour, Ahmed Intissar, Zouhaïr Mouayn (1997)
computed the reproducing kernel of this space:

$$K_z^{\mathcal{F}_m}(w) = e^{\overline{z}w} \, L_{m-1}^{(1)}(|w - z|^2).$$

Polyanalytic Steinwart–Hush–Scovel space

$\mathcal{S}_m :=$ m-analytic functions $\mathbb{C} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{S}_m} := \left(\frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 \exp\left(-2\,\mathrm{Im}(z)^2\right) \mathrm{d}\mu_2(z) \right)^{1/2} < +\infty.$$

Isometric isomorphism $U_{\mathcal{F}_m}^{\mathcal{S}_m} \colon \mathcal{F}_m \to \mathcal{S}_m$,

$$(U_{\mathcal{F}_m}^{\mathcal{S}_m} f)(z) := \mathrm{e}^{-z^2/2} f(z).$$

Reproducing kernel of $\mathcal{S}_m$:

$$K_z^{\mathcal{S}_m}(w) = \mathrm{e}^{-\frac{1}{2}(w-\overline{z})^2} L_{m-1}^{(1)}(|w-z|^2).$$

The original Steinwart–Hush–Scovel space on $\mathbb{C}^n$

Ingo Steinwart, Don Hush, Clint Scovel (2006).

Analytic functions on $\mathbb{C}^n$ such that

$$\frac{2^n \alpha^{2n}}{\pi^n} \int_{\mathbb{C}^n} |f(z)|^2 \exp\left(-4\alpha^2 \sum_{j=1}^{n} \operatorname{Im}(z_j)^2\right) \mathrm{d}\mu_{2n}(z) < +\infty.$$

Reproducing kernel:

$$\exp\left(-\alpha^2 \sum_{j=1}^{n} (w_j - \overline{z_j})^2\right).$$

Its restriction to $\mathbb{R}^n$ (the Gaussian kernel) is widely used in machine learning.

"Flattened poly-Fock space"

Isometric isomorphism $U_{\mathcal{F}_m}^{\mathcal{H}_m} \colon \mathcal{F}_m \to \mathcal{H}_m < L^2(\mathbb{R}^2)$,

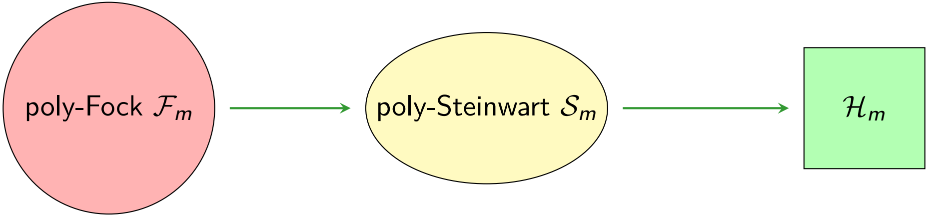
$$(U_{\mathcal{F}_m}^{\mathcal{H}_m} f)(x,y) := \left(\frac{2}{\pi}\right)^{1/4} \mathrm{e}^{-\frac{x^2+y^2}{2} - \mathrm{i}\, xy} f(x + \mathrm{i}\, y).$$

Reproducing kernel of $\mathcal{H}_m$:

$$K_{x,y}(u,v) = \sqrt{\frac{2}{\pi}}\, \mathrm{e}^{-\frac{(u-x)^2+(v-y)^2}{2} - \mathrm{i}\,(u-x)(v+y)}\, L_{m-1}^{(1)}((u-x)^2 + (v-y)^2).$$

We see that $\quad K_{x,y}(u,v) = K_{0,y}(u-x, v).$

$\mathcal{S}_m$ and $\mathcal{H}_m$ are "flattened" versions of the poly-Fock space $\mathcal{F}_m$



Weyl operators and horizontal translations

Unitary representation of $\mathbb{R}$ in $\mathcal{F}_m$:

$$(\rho_{\mathcal{F}_m}(a)f)(z) := f(z-a)\ e^{az-\frac{a^2}{2}}\ .$$

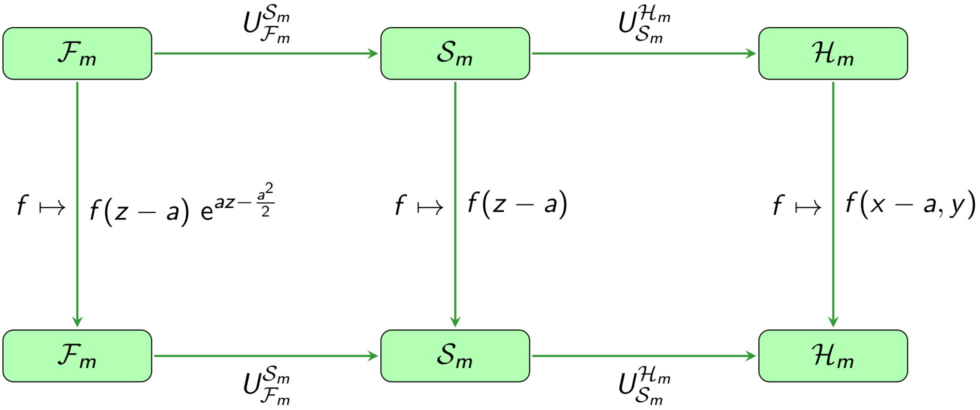
Unitary representation of $\mathbb{R}$ in $\mathcal{S}_m$:

$$(\rho_{\mathcal{S}_m}(a)f)(z) := f(z-a).$$

Unitary representation of $\mathbb{R}$ in $\mathcal{H}_m$:

$$(\rho_{\mathcal{H}_m}(a)f)(x,y) := f(x-a,y).$$

Three Hilbert spaces and corresponding unitary representations of $\mathbb{R}$



Fourier connection between Laguerre and Hermite functions

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\, u\xi}\, \mathrm{e}^{-\frac{u^2}{2}}\, L_n(u^2+a^2)\,\mathrm{d}u = \frac{1}{2^n\, n!}\, \mathrm{e}^{-\frac{\xi^2}{2}}\, H_n\left(\frac{\xi+a}{\sqrt{2}}\right) H_n\left(\frac{\xi-a}{\sqrt{2}}\right).$$

Equivalently,

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\, u\xi}\, \mathrm{e}^{-\frac{u^2+a^2}{2}}\, L_n(u^2+a^2)\,\mathrm{d}u = \sqrt{\pi}\, \psi_n\left(\frac{\xi+a}{\sqrt{2}}\right) \psi_n\left(\frac{\xi-a}{\sqrt{2}}\right).$$

Here ψ_n is the nth Hermite function: $\psi_n(t) = \dfrac{1}{\sqrt{2^n\, n!\, \sqrt{\pi}}}\, \mathrm{e}^{-t^2/2}\, H_n(t).$

Gerald B. Folland (1989). Harmonic Analysis on Phase Space.

Fourier transform of the reproducing kernel of $\mathcal{H}_m$

Proposition

For every ξ, y, v in $\mathbb{R}$,

$$L_{\xi,y}(v) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} K_{0,y}(u,v)\, \mathrm{e}^{-\mathrm{i}\, u\xi}\, \mathrm{d}u = \sum_{j=0}^{m-1} \overline{q_{j,\xi}(y)} q_{j,\xi}(v),$$

where

$$q_{j,\xi}(v) := 2^{1/4}\, \psi_j\left(\frac{\xi + 2v}{\sqrt{2}}\right).$$

Conclusions

$\Omega = \mathbb{R}$.

For every ξ in $\mathbb{R}$, $d_\xi = \dim(\widehat{H}_\xi) = m$.

For every ξ in $\mathbb{R}$, $\left(q_{0,\xi},\ q_{1,\xi},\ \ldots,\ q_{m-1,\xi}\right)$ is an orthonormal basis of $\widehat{H}_\xi$.

$\mathcal{C}(\rho_{\mathcal{F}_m}) \cong \mathcal{C}(\rho_{\mathcal{S}_m}) \cong \mathcal{C}(\rho_{\mathcal{H}_m}) = \mathcal{V} \cong L^\infty(\mathbb{R}, \mathcal{M}_n(\mathbb{C})) \cong L^\infty(\mathbb{R}) \otimes \mathcal{M}_n(\mathbb{C})$.

Vertical operators in $\mathcal{F}_m$ $\cong$ bounded matrix-functions on $\mathbb{R}$.

Possible themes for future works

- Vertical operators in $\mathcal{F}_m(\mathbb{C}^n)$.
- Angular operators in the wavelet spaces.
- More generally, shift-invariant operators associated to coherent states.
- Quasi-hyperbolic and quasi-nilpotent case in $\mathcal{A}^2(\mathbb{B}_n, \mu_{n,\alpha})$.
- Compute directly the Fourier transform of the reproducing kernel of the m-poly-Bergman space on the upper half-plane.
- Radial operators in many RKHS over the unit disk.