Horizontal Fourier transform of the polyanalytic Fock kernel

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Outline

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1. Problem
2. Laguerre and Hermite functions
3. Transformations of the kernel
4. Main results

Wirtinger operator

For $k = (k_1, \ldots, k_n)$, $k_1, \ldots, k_n \in \{0, 1, 2, \ldots\}$,

$$\overline{D}^k := \frac{\partial^{|k|}}{\partial \overline{z_1}^{k_1} \ldots \partial \overline{z_n}^{k_n}}.$$

m-analytic functions on $\mathbb{C}^n$

Let $m \in \mathbb{N}$ and $f\colon \mathbb{C}^n \to \mathbb{C}$ be a smooth function.

The following conditions are equivalent:

(a) $\overline{D}^k f \equiv 0 \quad$ for all $k = (k_1, \ldots, k_n)$ with $k_1 + \cdots + k_n = m$;

(b) $f(z) = \sum_{j_1+\ldots+j_n \le m-1} g_j(z)\,\overline{z}^j \quad$ for some analytic functions g_j.

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Leal-Pacheco, Maximenko, Ramos-Vazquez (2021):
Homogeneously polyanalytic kernels on the unit ball and the Siegel domain.

m-analytic Fock space on $\mathbb{C}^n$

$\mu :=$ Lebesgue measure.

Gaussian measure on $\mathbb{C}^n$:

$$\mathrm{dG} := \frac{1}{\pi^n}\,\mathrm{e}^{-|z|^2}\,\mathrm{d}\mu.$$

More generally, $\mathrm{dG} = \dfrac{\alpha^n}{\pi^n}\,\mathrm{e}^{-\alpha|z|^2}\,\mathrm{d}\mu.$

$$\|f\|_{\mathcal{F}} := \left(\int_{\mathbb{C}^n} |f|^2\,\mathrm{dG}\right)^{1/2}.$$

$$\mathcal{F} := \left\{f\colon \mathbb{C}^n \to \mathbb{C}\colon \quad f \text{ is } m\text{-analytic},\ \|f\|_{\mathcal{F}} < +\infty\right\}.$$

Reproducing kernel

$$K_z^{\mathcal{F}}(w) = \mathrm{e}^{\langle w,z\rangle}\, L_{m-1}^{(n)}(|w-z|^2).$$

Youssfi (2021):
Polyanalytic reproducing kernels in $\mathbb{C}^n$.

Weighted translations (Weyl operators)

For every a in $\mathbb{C}^n$, the Weyl operator acts on $L^2(\mathbb{C}^n, \mathrm{G})$ by

$$(W_a f)(z) := f(z-a)\,\mathrm{e}^{\langle z,a\rangle - \frac{1}{2}|a|^2}.$$

In the present work, we suppose $a \in \mathbb{R}^n$
and consider the compressions of W_a to $\mathcal{F}$:

$$\rho(a)\colon \mathcal{F} \to \mathcal{F}, \qquad \rho(a)f := W_a f.$$

$(\rho, \mathcal{F})$ is a unitary representation of $\mathbb{R}^n$.

Main object of study

The centralizer of ρ:

$$\mathcal{C}(\rho) := \Big\{ S \in \mathcal{B}(\mathcal{F}) : \quad \forall a \in \mathbb{R}^n \quad S\,\rho(a) = \rho(a)\,S \Big\}.$$

Our study is inspired by

- Vasilevski (2000):
  Poly-Fock spaces.

- Arroyo-Neri, Sánchez-Nungaray, Hernández-Marroquin, López-Martínez (2021):
  Toeplitz operators with Lagrangian invariant symbols acting on the poly-Fock space of $\mathbb{C}^n$.

Main difference:
now we apply the Fourier transform to the reproducing kernel, not to the differential equations.

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Laguerre and Hermite polynomials

Laguerre(–Sonin) polynomials:

$$L_n^{(\alpha)}(x) := \frac{x^{-\alpha}\,\mathrm{e}^{x}}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x}\,x^{n+\alpha}\right).$$

Hermite polynomials:

$$H_n(x) := (-1)^n\,\mathrm{e}^{x^2}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x^2}).$$

Decomposition formula for Laguerre polynomials

$$L_p^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{p} L_k^{(\alpha)}(x)\, L_{p-k}^{(\beta)}(y).$$

By induction,

$$L_p^{(n)}(t_1+\cdots+t_n) = \sum_{k_1+\ldots+k_n\leq p} \prod_{r=1}^{n} L_{k_r}(t_r).$$

Laguerre and Hermite functions

Laguerre functions form an orthonormal basis of $L^2(\mathbb{R}_+)$:

$$\ell_m(t) := \mathrm{e}^{-\frac{1}{2}t} L_m(t).$$

Hermite functions form an orthonormal basis of $L^2(\mathbb{R})$:

$$\psi_n(t) := (2^n\, n!\, \sqrt{\pi})^{-\frac{1}{2}}\, \mathrm{e}^{-\frac{1}{2}t^2} H_n(t).$$

Fourier connection between Laguerre and Hermite functions

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\,u\xi}\,\ell_n(u^2+a^2)\,\mathrm{d}u = \sqrt{\pi}\,\psi_n\left(\frac{\xi+a}{\sqrt{2}}\right)\psi_n\left(\frac{\xi-a}{\sqrt{2}}\right).$$

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This is equivalent to "Wigner distributions of Hermite functions".

Groenewold (1946):

On the principles of elementary quantum mechanics.

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Groenewold (1946):
On the principles of elementary quantum mechanics.

Folland (1989):
Harmonic Analysis in Phase Space.

Thangavelu (1993):
Lectures on Hermite and Laguerre Expansions.

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Multiplication by weight

$$\mathcal{F} \le L^2(\mathbb{C}^n, \mathrm{G}) \xrightarrow{\quad U \quad} \mathcal{H} \le L^2(\mathbb{R}^{2n})$$

U is a unitary operator defined by

$$(Uf)(u,v) := 2^{\frac{n}{2}}\, \mathrm{e}^{-\frac{|u|^2+|v|^2}{2} - \mathrm{i}\,\langle u,v\rangle}\, f(u+\mathrm{i}\,v).$$

The reproducing kernel of $\mathcal{H}$ is

$$K^{\mathcal{H}}_{x,y}(u,v) = 2^n\, \mathrm{e}^{-\frac{|u-x|^2+|v-y|^2}{2} - \mathrm{i}\,\langle u-x, v+y\rangle}\, L^{(n)}_{m-1}(|u-x|^2 + |v-y|^2).$$

Decomposition of the reproducing kernel of $\mathcal{H}$

$$K^{\mathcal{H}}_{x,y}(u,v) = 2^n \sum_{|k|\le m-1} \prod_{r=1}^{n} \mathrm{e}^{-\mathrm{i}\,(u_r - x_r)(v_r + y_r)}\ \ell_{k_r}\big((u_r - x_r)^2 + (v_r - y_r)^2\big).$$

In each term, the variables are separated:
the factor with index r depends on u_r, v_r, x_r, y_r.

Fourier transform of the reproducing kernel

$$L_{\xi,y}(v) := \int_{\mathbb{R}^n} K^{\mathcal{H}}_{0,y}(u,v)\, \mathrm{e}^{-\mathrm{i}\langle u,\xi\rangle}\, \mathrm{d}\widetilde{\mu}(u), \qquad \widetilde{\mu} := \frac{1}{(2\pi)^{\frac{n}{2}}}\mu.$$

Theorem

$$L_{\xi,y}(v) = \sum_{|k|\le m-1} q_{\xi,k}(y) q_{\xi,k}(v),$$

where

$$q_{\xi,k}(v) := 2^{n/2}\pi^{n/4} \prod_{r=1}^{n} \psi_{k_r}\left(\frac{\xi_r + 2v_r}{\sqrt{2}}\right).$$

Outline

Main results

Herrera-Yañez, Maximenko, Ramos-Vazquez (2022):
Translation-invariant operators in RKHS.

According to the scheme from this paper,

$$\widehat{\mathcal{H}} := (F \otimes I)\mathcal{H} = \int_{\mathbb{R}^n}^{\oplus} X_\xi \, \mathrm{d}\widetilde{\mu}(\xi),$$

where X_ξ is a finite-dimensional RKHS with kernel $L_{\xi,y}(v)$,

with ON basis $\big(q_{\xi,k}\big)_{|k|\le m-1}$ and dimension $d := \binom{n+m-1}{n}$.

Isometric isomorphism N

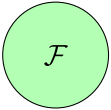
Define $N\colon \widehat{\mathcal{H}} \to L^2(\mathbb{R}^n)^d$,

$$(Ng)_k(\xi) := \langle g(\xi,\cdot), q_{\xi,k} \rangle.$$

So,

$$\widehat{\mathcal{H}} \cong \int_{\mathbb{R}^n}^{\oplus} \mathbb{C}^d \, \mathrm{d}\widetilde{\mu}(\xi) = L^2(\mathbb{R}^n, \mathbb{C}^d) = L^2(\mathbb{R}^n) \otimes \mathbb{C}^d = L^2(\mathbb{R}^n)^d.$$

Isometric isomorphisms between the spaces



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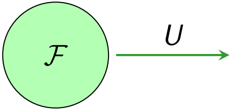

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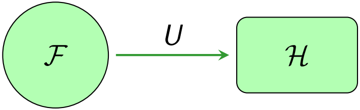

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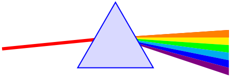

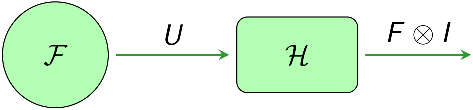

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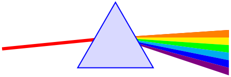

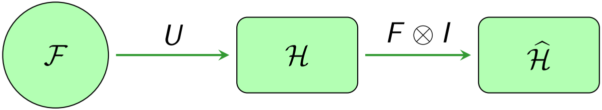

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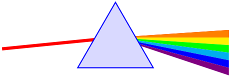

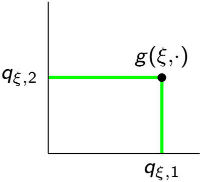

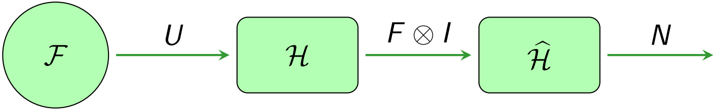

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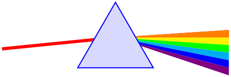

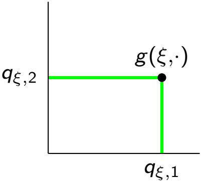

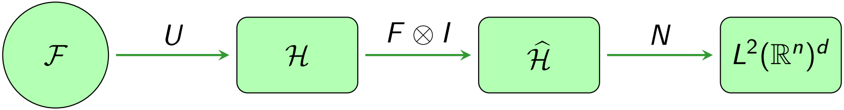

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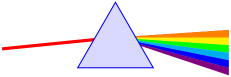
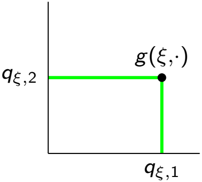
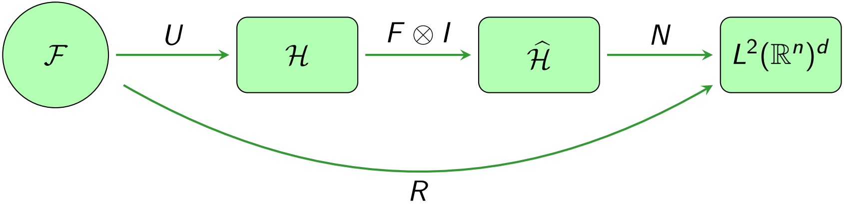
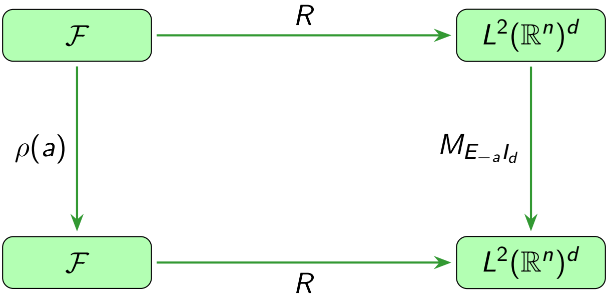






R intertwines Weyl translations with mult. by characters



$$E_{-a}(\xi) = \mathrm{e}^{-\mathrm{i}\,\langle \xi, a \rangle}\,.$$

Decomposition of $\mathcal{C}(\rho)$

R intertwines the translation-invariant operators with multiplications by matrix-functions.

Theorem

$$R\,\mathcal{C}(\rho)\,R^* \;\cong\; L^\infty(\mathbb{R}^n)^{d\times d}.$$

$$d = \binom{n+m-1}{n}.$$

For $m \geq 2$, this algebra is not commutative.

From vertical Toeplitz operators to matrix-functions

In particular, let $g \in L^\infty(\mathbb{R}^n)$ and

$$\widetilde{g}(u + \mathrm{i}\, v) := g(v).$$

Then the Toeplitz operator $T_{\widetilde{g}}$ acting in $\mathcal{F}$ belongs to $\mathcal{C}(\rho)$, and

$$R T_{\widetilde{g}} R^* = M_{\gamma_g},$$

where

$$\gamma_g(\xi) := \left[\int_{\mathbb{R}^n} g(v)\, \overline{q_{r,\xi}(v)}\, q_{s,\xi}(v)\, \mathrm{d}\widetilde{\mu}(v) \right]_{|r|,|s| \le m-1}.$$