Inverses of positive tridiagonal Toeplitz matrices

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Abstract

This simple note contains exact and approximate formulas for the inverse of the matrix $T_n+\alpha I_n$, where T_n is the tridiagonal real symmetric Toeplitz matrix of order n with entries $-1, 2, -1$; I_n is the identity matrix of order n, and $\alpha > 0$. General formulas for the inverses of banded Toeplitz matrices were deduced by Trench; many other authors considered particular cases. In this text we just consider one example in a very detailed manner.

Given numbers $\alpha > 0$ and $n \in \{1, 2, 3, 4, \ldots\}$, in this text we denote by T_n the $n \times n$ Toeplitz matrix generated by the symbol

$$a(t) = -t^{-1} + 2 - t,$$

or, after the change of variable $t = e^{i\theta}$,

$$g(\theta) = 4\left(\sin\frac{\theta}{2}\right)^2.$$

The first column of the matrix T_n is $[2, -1, 0, \ldots, 0]^\top$. For example,

$$T_6 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

The matrix T_n naturally appears in the discretized Laplace–Dirichlet problem on a segment. In other words, T_n is the grounded Laplace matrix (i.e. the Laplace–Dirichlet matrix) associated to the path graph, with grounded extreme vertices. The matrix T_n is also known as the favorite matrix of Gilbert Strang.

Furthermore, consider the matrices $T_n + \alpha I_n$, where I_n is the $n \times n$ identity matrix and $\alpha > 0$. For example,

$$T_6 + \alpha I_6 = \begin{bmatrix} 2+\alpha & -1 & 0 & 0 & 0 & 0 \\ -1 & 2+\alpha & -1 & 0 & 0 & 0 \\ 0 & -1 & 2+\alpha & -1 & 0 & 0 \\ 0 & 0 & -1 & 2+\alpha & -1 & 0 \\ 0 & 0 & 0 & -1 & 2+\alpha & -1 \\ 0 & 0 & 0 & 0 & -1 & 2+\alpha \end{bmatrix}.$$

The matrix T_n is positive definite, therefore $T_n+\alpha I_n$ is also positive definite. Consequently, $T_n + \alpha I_n$ is invertible. General formulas for the inverses of banded Toeplitz matrices were found by Trench [1]. Many other authors considered particular cases. In this text we present various equivalent explicit formulas for $(T_n + \alpha I_n)^{-1}$. All these formulas can be deduced from [1]. See also our interactive visualization [2].

Hyperbolic change of variable

It is convenient to write $2 + \alpha$ as $2\cosh(\beta)$, with $\beta > 0$. In other words, α and β are related by

$$\alpha = 4\left(\sinh\frac{\beta}{2}\right)^2, \qquad \beta = 2\operatorname{arcsinh}\frac{\sqrt{\alpha}}{2} = 2\ln\left(\frac{\sqrt{\alpha}}{2} + \sqrt{1+\frac{\alpha}{4}}\right).$$

The generating symbol of the Toeplitz matrix $T_n + \alpha I_n$ is

$$\alpha + g(\theta) = 4\left(\sinh\frac{\beta}{2}\right)^2 + 4\left(\sin\frac{\theta}{2}\right)^2. \tag{1}$$

Explicit formulas for the entries of $(T_n + \alpha I_n)^{-1}$

For every $j, k \in \{0, 1, \ldots, n-1\}$, the (j, k)-st entry of $(T_n + \alpha I_n)^{-1}$ equals

$$((T_n + \alpha I_n)^{-1})_{j,k} = \begin{cases} \dfrac{\sinh((k+1)\beta)\sinh((n-j)\beta)}{\sinh(\beta)\sinh((n+1)\beta)}, & j \ge k; \\[2ex] \dfrac{\sinh((j+1)\beta)\sinh((n-k)\beta)}{\sinh(\beta)\sinh((n+1)\beta)}, & j < k. \end{cases} \tag{2}$$

Two cases can be joined in the following manner:

$$((T_n + \alpha I_n)^{-1})_{j,k} = \frac{\sinh((\min(j,k)+1)\beta)\sinh((n-\max(j,k))\beta)}{\sinh(\beta)\sinh((n+1)\beta)}. \tag{3}$$

The product of sinh in the numerator can be transformed into a difference of cosh. Namely, for every $j, k \in \{0, 1, \ldots, n-1\}$,

$$((T_n + \alpha I_n)^{-1})_{j,k} = \frac{\cosh((n+1-|j-k|)\beta) - \cosh(|n-1-(j+k)|\beta)}{2\sinh(\beta)\sinh((n+1)\beta)}. \tag{4}$$

$(T_n + \alpha I_n)^{-1}$ as Toeplitz matrix minus Hankel matrix

It follows from (4) that

$$(T_n + \alpha I_n)^{-1} = S_{\alpha,n} - H_{\alpha,n}, \tag{5}$$

where $S_{\alpha,n}$ and $H_{\alpha,n}$ are matrices with the following entries ($j, k \in \{0, 1, \ldots, n-1\}$):

$$(S_{\alpha,n})_{j,k} = \frac{\cosh((n+1-|j-k|)\beta)}{2\sinh(\beta)\sinh((n+1)\beta)}, \qquad (H_{\alpha,n})_{j,k} = \frac{\cosh(|n-1-(j+k)|\beta)}{2\sinh(\beta)\sinh((n+1)\beta)}.$$

In other words, $S_{\alpha,n}$ is the symmetric Toeplitz matrix with 0-st column

$$\frac{1}{2\sinh(\beta)\sinh((n+1)\beta)} \left[\cosh((n+1-j)\beta)\right]_{j=0}^{n-1},$$

and $H_{\alpha,n}$ is the persymmetric Hankel matrix with 0-st column

$$\frac{1}{2\sinh(\beta)\sinh((n+1)\beta)} \left[\cosh((n-1-j)\beta)\right]_{j=0}^{n-1}.$$

Relation with Kac–Murdock–Szegő family of Toeplitz matrices

Given $\rho \in (0, 1)$, consider the $n \times n$ Toeplitz matrix

$$\mathrm{KMS}_{\rho,n} = \left[\rho^{-|j-k|}\right]_{j,k=0}^{n-1}.$$

The matrices $\mathrm{KMS}_{\rho,n}$ are known as Kac–Murdock–Szegő Toeplitz matrices. In this text we always assume that ρ is related with α and β by

$$\rho = e^{-\beta}, \qquad 2 + \alpha = \rho + \frac{1}{\rho}, \qquad \rho = \frac{1}{1 + \frac{\alpha}{2} + \sqrt{\alpha + \frac{\alpha^2}{4}}}.$$

The matrices $\mathrm{KMS}_{\rho,n}$ are generated by the symbol

$$\sum_{k=-\infty}^{+\infty} \rho^{-|k|} t^k = \sum_{k=-\infty}^{+\infty} \rho^{-|k|} e^{k\theta} = \frac{1-\rho^2}{1 - 2\rho\cos(\theta) + \rho^2}$$
$$= \frac{2\sinh(\beta)}{4\left(\sinh\frac{\beta}{2}\right)^2 + 4\left(\sin\frac{\theta}{2}\right)^2} = \frac{2\sinh(\beta)}{g(\theta) + \alpha}.$$

Recall that $g(\theta) + \alpha$ is the generating symbol of $T_n + \alpha I_n$, see (1).

If the parameter α is fixed and n is large enough, then the matrix $S_{\alpha,n}$ from (5) is very close to the matrix $\frac{1}{2\sinh(\beta)}\mathrm{KMS}_{\rho,n}$:

$$S_{\alpha,n} \approx \frac{1}{2\sinh(\beta)} \mathrm{KMS}_{\rho,n}. \tag{6}$$

Note that the entries of the matrix in the right-hand side of (6) don't depend on n.

Here is a precise version of (6). For every $\alpha > 0$, every $n \in \{1, 2, \ldots\}$, and every $j, k \in \{0, 1, \ldots, n-1\}$,

$$0 \le \left(S_{\alpha,n} - \frac{1}{2\sinh(\beta)} \operatorname{KMS}_{\rho,n} \right)_{j,k} \le \frac{1}{2\sinh(\beta)\sinh((n+1)\beta}. \tag{7}$$

In fact,

$$\begin{aligned} \left(S_{\alpha,n} - \frac{1}{2\sinh(\beta)} \operatorname{KMS}_{\rho,n} \right)_{j,k} &= \frac{1}{2\sinh(\beta)} \left(\frac{\cosh((n+1-d)\beta)}{\sinh((n+1)\beta} - e^{-d\beta} \right) \\ &= \frac{1}{2\sinh(\beta)} \frac{e^{-(n+1-d)\beta}}{\sinh((n+1)\beta} \le \frac{1}{2\sinh(\beta)\sinh((n+1)\beta)}. \end{aligned}$$

If β is fixed and n tends to infinity, then the last expression decays exponentially and uniformly in j and k.

The entries of the Hankel matrix $H_{\alpha,n}$ also can be approximated by exponentials. Moreover, this Hankel matrix is "concentrated" near the upper–left and bottom–right corners.

Numerical test

The following code in MATLAB language allows to construct the matrix $T_n + \alpha I_n$. This code was tested in GNU Octave.

```
function [] = testtoeplitztridiagonal(),
   al = 1.234567; n = 5;
   T = ToeplitzTridiagonal(al, n);
   Tinv = InverseOfToeplitzTridiagonal(al, n);
   disp('T ='); display(T);
   disp('Tinv ='); display(Tinv);
   disp('T * Tinv = '); display(T * Tinv);
   disp('norm(T * Tinv - eye(n)) = '); display(norm(T * Tinv - eye(n)));
end

function [T] = ToeplitzTridiagonal(al, n),
   col = [2 + al; -1; zeros(n - 2, 1)];
   T = toeplitz(col);
end

function [Tinv] = InverseOfToeplitzTridiagonal(al, n),
   be = 2 * log(sqrt(al) / 2 + sqrt(1 + al / 4));
   coef = 1 / (2 * sinh(be) * sinh((n + 1) * be));
   ind = (0 : n - 1)';
   coltoeplitz = coef * cosh((n + 1 - ind) * be);
   colhankel = coef * cosh((n - 1 - ind) * be);
   S = toeplitz(coltoeplitz);
   H = hankel(colhankel, colhankel(n : -1 : 1));
   Tinv = S - H;
end
```

References

[1] Trench, William F. (1985): Explicit inversion formulas for Toeplitz band matrices. SIAM. J. on Algebraic and Discrete Methods, 6:4, 546–554. DOI: 10.1137/0606054

[2] Maximenko, Egor A.; Sánchez Arzate, Gabino (2016): The inverse of the positive definite symmetric tridiagonal Toeplitz matrix, interactive visualization. http://www.egormaximenko.com/plots/tp3inverse.html