Vertical Toeplitz operators on the upper halfplane and logarithmically oscillating functions

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Bergman space on the upper halfplane

$$\Pi := \left\{ z \in \mathbb{C} : \ \ \operatorname{Im}(z) > 0 \right\}.$$

$\mathcal{A}(\Pi)$:= the vector space of all analytic functions on Π.

μ_2 := Lebesgue measure on the plane.

$$\mathcal{A}^2(\Pi) := \left\{ f \in \mathcal{A}(\Pi) : \quad \|f\|_{L^2(\Pi,\mu_2)} < +\infty \right\}.$$

$P\colon L^2(\Pi, \mu_2) \to L^2(\Pi, \mu_2)$ orthogonal projection whose image is $\mathcal{A}^2(\Pi)$.

Vertical Toeplitz operators

$\mathbb{R}_+ := (0, +\infty)$.

Given a in $L^\infty(\mathbb{R}_+)$, define A in $L^\infty(\Pi)$ by

$$A(z) := a(\mathrm{Im}(z)).$$

$T_a :=$ Toeplitz operator acting in $\mathcal{A}^2(\Pi)$ by

$$T_a f := P(A\, f).$$

Isometric isomorphism that diagonalizes all vertical operators

Isometric isomorphism $R\colon \mathcal{A}^2(\Pi) \to L^2(\mathbb{R}_+)$,

$$(R\,f)(\xi) := \frac{\sqrt{x}}{\sqrt{\pi}} \int_{\Pi} f(w)\, \mathrm{e}^{-\mathrm{i}\,\overline{w}\,\xi}\, \mathrm{d}\mu_2(w).$$

Its inverse:

$$(R^* g)(w) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} g(\xi)\, \sqrt{\xi}\, \mathrm{e}^{\mathrm{i}\,w\,\xi}\, \mathrm{d}\xi.$$

Diagonalization of vertical Toeplitz operators

For every a in $L^\infty(\mathbb{R}_+)$, $RT_a R^* = M_{\gamma_a}$, where

$$\gamma_a(\xi) := 2\xi \int_0^{+\infty} a(t)\, e^{-2\xi t}\, \mathrm{d}t.$$

It is easy to see that $\gamma_a \in C_b(\mathbb{R}_+)$.

Vasilevski (1999):
On the structure of Bergman and poly-Bergman spaces.

Natural question (by Nikolai Vasilevski)

Describe the C*-subalgebra of $C_b(\mathbb{R}_+)$ generated by

$$G := \Big\{\gamma_a\colon \ a \in L^\infty(\mathbb{R}_+)\Big\}.$$

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Log-distance on the positive halfline

Define $\rho\colon \mathbb{R}_+^2 \to [0, +\infty)$,

$$\rho(x, y) := \Big|\log(x) - \log(y)\Big|.$$

So, log is an isometry from $(\mathbb{R}_+, \rho)$ onto $(\mathbb{R}, d_{\mathbb{R}})$.

Measurer of uniform continuity with respect to ρ

"Modulus of continuity with respect to ρ"

Given f in $C_b(\mathbb{R}_+)$ and $\delta > 0$,

$$\omega_f(\delta) := \sup\left\{|f(x) - f(y)|: \quad x, y \in \mathbb{R}_+, \quad \rho(x, y) \le \delta\right\}.$$

Log-oscillating functions

"Very slowly oscillating functions"

Bounded functions $\mathbb{R}_+ \to \mathbb{C}$ that are uniformly continuous wrt ρ:

$$C_{b,u}(\mathbb{R}_+, \rho) := \left\{ f \in C_b(\mathbb{R}_+) : \quad \lim_{\delta \to 0} \omega_f(\delta) = 0 \right\}.$$

$$f \in C_{b,u}(\mathbb{R}_+, \rho) \quad \Longleftrightarrow \quad f \circ \exp \in C_{b,u}(\mathbb{R}).$$

$C_{b,u}(\mathbb{R}_+, \rho)$ is a C*-subalgebra of $C_b(\mathbb{R}_+)$.

Proposition (functions γ_a are log-oscillating)

Let $a \in L^\infty(\mathbb{R}_+)$. Then $\gamma_a \in C_{b,u}(\mathbb{R}_+, \rho)$.

More precisely, γ_a is Lipschitz-continuous with respect to ρ:

$$\omega_{\gamma_a}(\delta) \leq 2\,\delta.$$

Proposition (functions γ_a are log-oscillating)

Let $a \in L^\infty(\mathbb{R}_+)$. Then $\gamma_a \in C_{b,u}(\mathbb{R}_+, \rho)$.

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$$\omega_{\gamma_a}(\delta) \leq 2\,\delta.$$

Beginning of the proof (idea by Kevin Esmeral García):

$$|\gamma_a(x) - \gamma_a(y)| \leq \|a\|_\infty \int_0^{+\infty} \Big| \underbrace{2vx\ \mathrm{e}^{-2vx} - 2vy\ \mathrm{e}^{-2vy}}_{E_{x,y}(v)} \Big| \frac{\mathrm{d}v}{v}.$$

$E_{x,y}(v)$ changes its sign at $v_0 = \dfrac{\log(y) - \log(x)}{2(y-x)}$.

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1. Diagonalization of vertical Toeplitz operators
2. Log-oscillating functions
3. Main result

Main result

Recall that G is the set of the spectral functions:

$$G := \left\{\gamma_a\colon \ a \in L^\infty(\mathbb{R}_+)\right\}.$$

Theorem

G is a dense subset of $C_{b,u}(\mathbb{R}_+, \rho)$:

$$\operatorname{clos}_{C_b(\mathbb{R}_+)}(G) = C_{b,u}(\mathbb{R}_+, \rho).$$

Mellin convolution (convolution over $\mathbb{R}_+$)

$\mathbb{R}_+$ with the standard multiplication and usual topology is a locally compact abelian group.

The corresponding Haar measure ν is given by $\mathrm{d}\nu(x) = \frac{\mathrm{d}x}{x}$.

Given f, g in $L^1(\mathbb{R}_+, \nu)$,

$$(f * g)(x) := \int_0^{+\infty} f(y)\, g\left(\frac{x}{y}\right) \frac{\mathrm{d}y}{y}.$$

A Dirac sequence in $L^1(\mathbb{R}_+, \nu)$

$$\psi_n(s) := \frac{1}{\mathrm{B}(n,n)} \frac{s^n}{(1+s)^{2n}}.$$

Proposition

$(\psi_n)_{n\in\mathbb{N}}$ is a Dirac sequence:

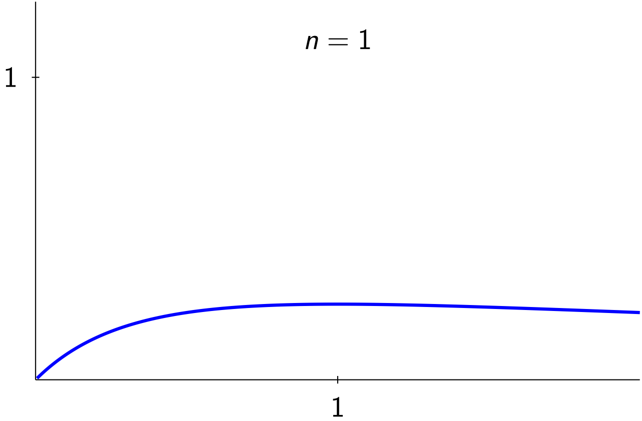
(1) $\psi_n \geq 0$;

(2) $\|\psi_n\|_{L^1(\mathbb{R}_+,\nu)} = 1$;

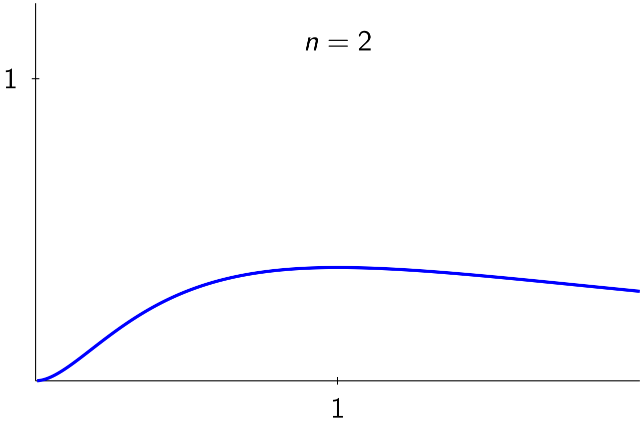
(3) for every $\delta > 0$,

$$\lim_{n\to\infty} \int_{\rho(s,1)>\delta} \psi_n(s) \frac{\mathrm{d}s}{s} = 0.$$

Plots of ψ_n



Plots of ψ_n



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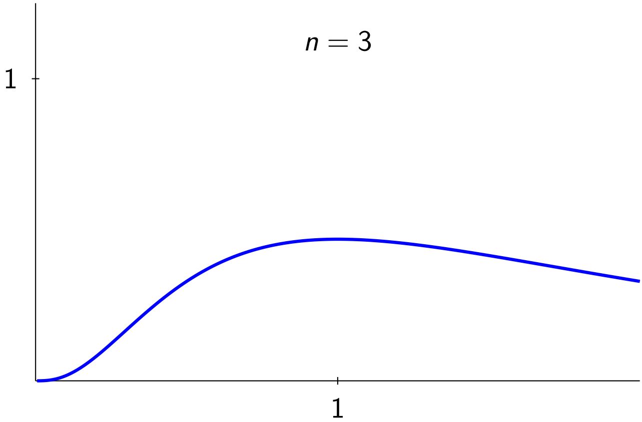

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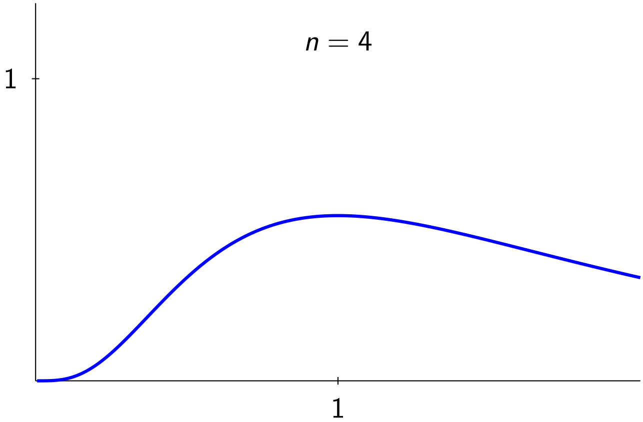

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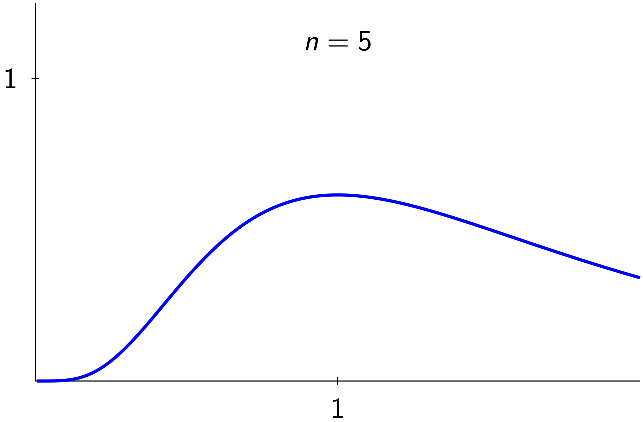

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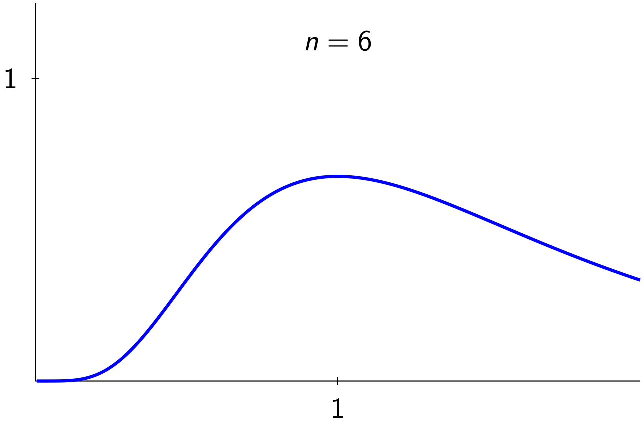

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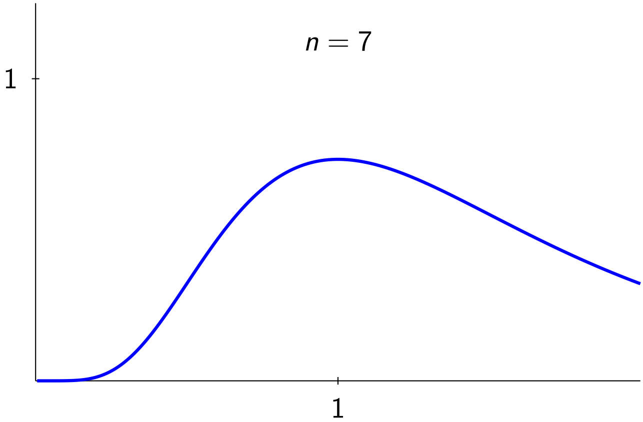

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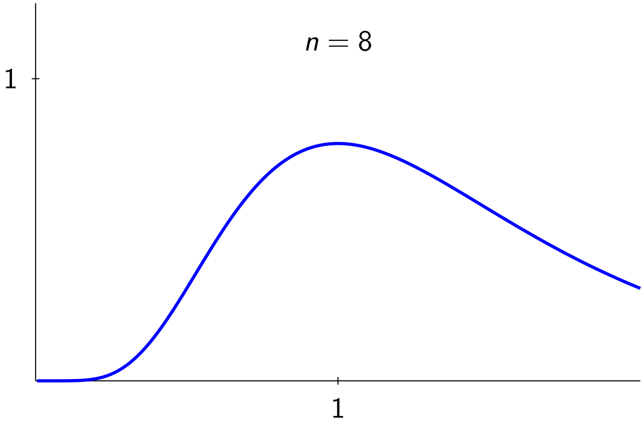

Elements of $C_{b,u}(\mathbb{R}_+, \rho)$ can be approximated by convolutions

Proposition

Let $\sigma \in C_{b,u}(\mathbb{R}_+, \rho)$. Then

$$\lim_{n\to\infty} \|\sigma * \psi_n - \sigma\|_\infty = 0.$$

It is a known fact about uniformly continuous functions and Dirac sequences.

Spectral functions as Mellin convolutions

$$\gamma_a(\xi) \coloneqq 2\xi \int_0^{+\infty} a(t)\, \mathrm{e}^{-2\xi t}\, \mathrm{d}t.$$

Notice that

$$\gamma_a = \widetilde{a} * \varkappa,$$

where $\widetilde{a}(t) \coloneqq a(1/t)$,

$$\varkappa(t) \coloneqq 2t\ \mathrm{e}^{-2t}.$$

Function $\varkappa$ is “approximately invertible” in $L^1(\mathbb{R}_+, \nu)$

$$\varkappa(t) := 2t\, e^{-2t} \qquad (t \in \mathbb{R}_+).$$

Proposition

There exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $L^1(\mathbb{R}_+, \nu)$ such that

$$\varphi_n * \varkappa = \psi_n.$$

We have found an explicit expression for φ_n in terms of Laguerre–Sonin polynomials.

Another proof: Wiener division theorem.

Proof of the density

Let $\sigma \in C_{b,u}(\mathbb{R}_+, \rho)$.

Define $a_n := \widetilde{\sigma * \varphi_n}$. Then

$$\gamma_{a_n} = \widetilde{a_n} * \varkappa = \sigma * \varphi_n * \varkappa = \sigma * \varphi_n \xrightarrow{C_b(\mathbb{R}_+)} \sigma.$$

Close results and applications

- Herrera Yañez, Ondrej Hutník, Maximenko (2014).
  Extension to the weighted case.

- Maribel Loaiza, Carmen Lozano (2013).
  For vertical Toeplitz operators over the harmonic Bergman space, the spectral functions are

$$\gamma_a^{\text{harm}}(\xi) := \gamma_a(|\xi|).$$

- Herrera Yañez, Maximenko, Vasilevski (2015).
  The spectral functions of radial Toeplitz operators can be obtained via a discretization of the vertical case.