Radial operators on the polyanalytic weighted Bergman space

Egor Maximenko, joint results with Roberto Moisés Barrera-Castelán,
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2021-05-05,
Harmonic Analysis Seminar, Louisiana State University (via Zoom)

This talk is based on two publications.

- Maximenko, E.A.; Tellería-Romero, A.M. (2020):
  Radial operators in polyanalytic Bargmann–Segal–Fock spaces.
  https://doi.org/10.1007/978-3-030-44651-2_18
- Barrera-Castelán, R.M.; Maximenko, E.A.; Ramos-Vazquez, G. (2021):
  Radial operators on polyanalytic weighted Bergman spaces.
  https://doi.org/10.1007/s40590-021-00348-w

Main difference between these two papers:

Laguerre polynomials for the Bargmann–Segal–Fock space,
shifted Jacobi polynomials for the weighted Bergman spaces.

Outline

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Polyanalytic functions

A function $f\colon \mathbb{D} \to \mathbb{C}$ is called n-analytic if $f \in C^n(\mathbb{D})$ and

$$\left(\frac{\partial}{\partial \overline{z}}\right)^n f(z) = 0.$$

Notation: $f \in A_n(\mathbb{D})$.

$$f \in A_n(\mathbb{D}) \qquad \Longleftrightarrow \qquad \exists g_0, \ldots, g_{n-1} \in A(\mathbb{D}) \qquad f(z) = \sum_{k=0}^{n-1} g_k(z)\,\overline{z}^k.$$

Balk (1991): Polyanalytic functions.

Polyanalytic Bergman space $\mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$

Let $\alpha > -1$. We consider $\mathbb{D}$ with the usual radial weight:

$$\mathrm{d}\mu_\alpha(z) := \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha\,\mathrm{d}\mu(z).$$

Short notation for the corresponding norm and inner product: $\|\cdot\|_2$, $\langle\cdot,\cdot\rangle$.

$$\mathcal{A}_n^2 := \mathcal{A}_n^2(\mathbb{D}, \mu_\alpha) := \text{“}\mathcal{A}_n(\mathbb{D}) \cap L^2(\mathbb{D}, \mu_\alpha)\text{”} = \{f \in \mathcal{A}_n(\mathbb{D})\colon \ \|f\|_2 < +\infty\}.$$

In this talk, $\alpha > -1$ is fixed, and we will often supress the dependence on α.

$\mathcal{A}_n^2$ is a reproducing kernel Hilbert space.

Jacobi polynomials for the unit interval

"Shifted Jacobi polynomials" in terms of classical Jacobi polynomials:

$$Q_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(2x-1).$$

Rodrigues formula and explicit formula:

$$Q_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{n!}(1-x)^{-\alpha}x^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Bigl((1-x)^{n+\alpha}x^{n+\beta}\Bigr),$$

$$Q_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\binom{\alpha+\beta+n+k}{k}\binom{\beta+n}{n-k}(-1)^{n-k}x^k.$$

For $\alpha,\beta>-1$, $(Q_n^{(\alpha,\beta)})_{n=0}^{\infty}$ is orthogonal on $(0,1)$ with the weight $(1-x)^{\alpha}x^{\beta}$.

Reproducing property in polyanalytic Bergman spaces

Koshelev (1977): reproducing property for the unweighted case, $\alpha = 0$.

Pessoa (2014): nice and complete proof, connection with Jacobi polynomials.

Hachadi and Youssfi (2019): weighted case.

$$K_{n,z}^{(\alpha)}(w) = \frac{(1-\overline{w}z)^{n-1}}{(1-\overline{z}w)^{n+1}} R_{n-1}^{(\alpha,0)}\left(\left|\frac{z-w}{1-\overline{z}w}\right|^2\right),$$

$$R_n^{(\alpha,\beta)}(x) := \frac{(-1)^n \ \mathrm{B}(\alpha+1,\beta+1)}{\mathrm{B}(\alpha+n+1,\beta+1)} Q_n^{(\alpha,\beta+1)}(x).$$

Leal-Pacheco, Maximenko, Ramos-Vazquez (2021), independently Youssfi (2021): generalization to the unit ball in $\mathbb{C}^n$.

Monomials and their frequencies

Monomial functions $m_{p,q}(z) = z^p \, \overline{z}^q$ with $q < n$ are typical inhabitants of $\mathcal{A}_n^2$.
In polar coordinates: $z = r\,t$, $r \in [0,1)$, $t \in \mathbb{T}$,

$$m_{p,q}(z) = z^p \, \overline{z}^q = r^{p+q} t^{p-q}.$$

We say that $m_{p,q}$ has frequency $p - q$.

Monomials with different frequencies are orthogonal:

$$\langle m_{p,q}, m_{j,k} \rangle = (\alpha + 1) \, \mathrm{B}\,(p + j + 1, \alpha + 1)\; \delta_{p-q,j-k}.$$

Monomials and their frequencies

Different frequencies correspond to different diagonals in the following table:

$$
\begin{aligned}
&m_{0,0}(z) = z^0\overline{z}^0 = r^0 t^0, \quad m_{0,1}(z) = z^0\overline{z}^1 = r^1 t^{-1}, \quad m_{0,2}(z) = z^0\overline{z}^2 = r^2 t^{-2},\\
&m_{1,0}(z) = z^1\overline{z}^0 = r^1 t^1, \quad m_{1,1}(z) = z^1\overline{z}^1 = r^2 t^0, \quad m_{1,2}(z) = z^1\overline{z}^2 = r^3 t^{-1},\\
&m_{2,0}(z) = z^2\overline{z}^0 = r^2 t^2, \quad m_{2,1}(z) = z^2\overline{z}^1 = r^3 t^1, \quad m_{2,2}(z) = z^2\overline{z}^2 = r^4 t^0.
\end{aligned}
$$

Disk polynomials, also called Jacobi polynomials in z and $\overline{z}$

Koornwinder (1975), Wünsche (2005).

$$b_{p,q}^{(\alpha)}(z) := (-1)^{p+q}\, c_{\alpha,p,q}\, (1 - z\overline{z})^{-\alpha}\, \frac{\partial^q}{\partial z^q}\, \frac{\partial^p}{\partial \overline{z}^p}\Big((1 - z\,\overline{z})^{p+q+\alpha}\Big).$$

For brevity, we will write just $b_{p,q}$.

Expression in terms of shifted Jacobi polynomials:

$$b_{p,q}(r\,t) = \widetilde{c}_{\alpha,p,q}\, t^{p-q}\, r^{|p-q|}\, Q_{\min\{p,q\}}^{(\alpha,|p-q|)}(r^2) \qquad (0 \le r < 1,\ t \in \mathbb{T}).$$

$(b_{p,q})_{p,q=0}^{\infty}$ is an orthonormal basis of $L^2(\mathbb{D}, \mu_\alpha)$.

Orthonormal basis of $\mathcal{A}_n^2$

Proposition

$(b_{p,q})_{p\geq 0, 0\leq q<n}$ is an orthonormal basis of $\mathcal{A}_n^2$.

Orthonormal basis of $\mathcal{A}_n^2$

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Idea of the proof. Ramazanov (1999) for $\alpha = 0$.

We have to show that this family is total in $\mathcal{A}_n^2$.

Let $f \in \mathcal{A}_n^2$, $f \perp b_{p,q}$ for all p, q, and

$$f = \sum_{j=0}^{\infty} \sum_{k=0}^{n-1} c_{j,k} m_{j,k}.$$

The conditions $f \perp b_{p,q}$ yield a triangular system of homogeneous equations.

Therefore, all coefficients $c_{j,k}$ must be zero.

Generators of $\mathcal{A}^2_3$ and an orthonormal basis of $\mathcal{A}^2_3$:

$m_{0,0}$	$m_{0,1}$	$m_{0,2}$	$m_{0,3}$	$m_{0,4}$	$\ddots$
$m_{1,0}$	$m_{1,1}$	$m_{1,2}$	$m_{1,3}$	$m_{1,4}$	$\ddots$
$m_{2,0}$	$m_{2,1}$	$m_{2,2}$	$m_{2,3}$	$m_{2,4}$	$\ddots$
$m_{3,0}$	$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	$m_{3,4}$	$\ddots$
$m_{4,0}$	$m_{4,1}$	$m_{4,2}$	$m_{4,3}$	$m_{4,4}$	$\ddots$
$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

$b_{0,0}$	$b_{0,1}$	$b_{0,2}$	$b_{0,3}$	$b_{0,4}$	$\ddots$
$b_{1,0}$	$b_{1,1}$	$b_{1,2}$	$b_{1,3}$	$b_{1,4}$	$\ddots$
$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$b_{2,4}$	$\ddots$
$b_{3,0}$	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$	$b_{3,4}$	$\ddots$
$b_{4,0}$	$b_{4,1}$	$b_{4,2}$	$b_{4,3}$	$b_{4,4}$	$\ddots$
$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

True-n-analytic Bergman spaces

$$\mathcal{A}^2_{(n)} := \{f \in \mathcal{A}^2_n : \ f \perp \mathcal{A}^2_{n-1}\}.$$

$$\mathcal{A}^2_n = \bigoplus_{m=1}^{n} \mathcal{A}^2_{(m)}.$$

$$L^2(\mathbb{D}, \mu_\alpha) = \bigoplus_{m=1}^{\infty} \mathcal{A}^2_{(m)}.$$

Orthonormal basis of $\mathcal{A}^2_{(2)}$:

$$\begin{matrix} b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & b_{0,4} & \ddots \\ b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & \ddots \\ b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & \ddots \\ b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & \ddots \\ b_{4,0} & b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{matrix}$$

Positive results about Toeplitz operators in $\mathcal{A}_n^2$ and $\mathcal{A}_{(n)}^2$

The correspondence between bounded generating symbols and Toeplitz operators is injective:

$$\text{if} \quad a \in L^\infty(\mathbb{D}) \quad \text{such that} \quad T_{n,a} = 0, \quad \text{then} \quad a = 0 \quad \text{a.e.}$$

Proof: the same as in Berger and Coburn (1986).

Positive results about Toeplitz operators in $\mathcal{A}^2_n$ and $\mathcal{A}^2_{(n)}$

The correspondence between bounded generating symbols and Toeplitz operators is injective:

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Proof: the same as in Berger and Coburn (1986).

Rozenblum, Vasilevski (2019):
Toeplitz operators with bounded symbols in $\mathcal{A}^2_{(n)}$ are unitarily equivalent to Toeplitz operators with distributional symbols in $\mathcal{A}^2$.

A couple of exotic properties of $\mathcal{A}_n^2$

For $n \geq 2$, the Berezin transform for operators in $\mathcal{A}_n^2$ is not injective.

Engliš (2006) proved this fact for Bergman harmonic spaces. The proof is the same.

A couple of exotic properties of $\mathcal{A}_n^2$

Proposition

For $n \geq 2$, the set of all Toeplitz operators with bounded generating symbols

$$\{T_a\colon\ a \in L^\infty(\mathbb{D})\}$$

is not weakly dense in $\mathcal{B}(\mathcal{A}_n^2)$.

Idea of the proof. Let $f \in \mathcal{A}_n^2$ such that $\overline{f} \in \mathcal{A}_n^2$ and the functions $f, \overline{f}$ are lin. indep.

$$\{T_a\colon\ a \in L^\infty(\mathbb{D})\} \subseteq \underbrace{\{S \in \mathcal{B}(\mathcal{A}_n^2)\colon\ \langle Sf, f\rangle = \langle S\overline{f}, \overline{f}\rangle\}}_{\text{weakly closed, } \neq \mathcal{B}(\mathcal{A}_n^2)}.$$

Outline

Some of the following constructions are similar to ideas by Vasilevski.

Vasilevski (2008):

Commutative algebras of Toeplitz operators on the Bergman space.

Vasilevski applied the Fourier transform $\mathcal{F}_{\mathbb{T}}$ to the differential equation $\left(\frac{\partial}{\partial \overline{z}}\right)^n f = 0$.

We use the polynomial orthonormal basis.

Another possible way is to apply the Fourier transform $\mathcal{F}_{\mathbb{T}}$ to the reproducing kernel.

Truncated diagonal subspaces

For $\xi \in \mathbb{Z}$, $s \in \mathbb{N}$,

$$W_{\xi,s} := \operatorname{span}\{m_{p,q} \colon \ p - q = \xi, \ \min\{p, q\} < s\}, \qquad \dim(W_{\xi,s}) = s.$$

Disk polynomials form orthonormal bases of theses subspaces:

$$W_{\xi,s} := \operatorname{span}\{b_{p,q} \colon \ p - q = \xi, \ \min\{p, q\} < s\}.$$

Idea of the proof:

$$b_{p,q}(z) = \sum_{k=0}^{\min\{p,q\}} C_{\alpha,p,q,k} \ z^{p-k} \, \overline{z}^{q-k}.$$

Truncated diagonal subspaces

Generators of $W_{1,4}$ and an orthonormal basis of $W_{1,4}$:

$$
\begin{matrix}
m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & m_{0,4} & \ddots \\
\colorbox{lavender}{$m_{1,0}$} & m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & \ddots \\
m_{2,0} & \colorbox{lavender}{$m_{2,1}$} & m_{2,2} & m_{2,3} & m_{2,4} & \ddots \\
m_{3,0} & m_{3,1} & \colorbox{lavender}{$m_{3,2}$} & m_{3,3} & m_{3,4} & \ddots \\
m_{4,0} & m_{4,1} & m_{4,2} & \colorbox{lavender}{$m_{4,3}$} & m_{4,4} & \ddots \\
m_{5,0} & m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{matrix}
\qquad
\begin{matrix}
b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & b_{0,4} & \ddots \\
\colorbox{lavender}{$b_{1,0}$} & b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & \ddots \\
b_{2,0} & \colorbox{lavender}{$b_{2,1}$} & b_{2,2} & b_{2,3} & b_{2,4} & \ddots \\
b_{3,0} & b_{3,1} & \colorbox{lavender}{$b_{3,2}$} & b_{3,3} & b_{3,4} & \ddots \\
b_{4,0} & b_{4,1} & b_{4,2} & \colorbox{lavender}{$b_{4,3}$} & b_{4,4} & \ddots \\
b_{5,0} & b_{5,1} & b_{5,2} & b_{5,3} & b_{5,4} & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{matrix}
$$

Generators of $W_{-2,2}$ and an orthonormal basis of $W_{-2,2}$:

$$\begin{matrix} m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & m_{0,4} & \ddots \\ m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & \ddots \\ m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & \ddots \\ m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & \ddots \\ m_{4,0} & m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & \ddots \\ m_{5,0} & m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{matrix} \qquad \begin{matrix} b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & b_{0,4} & \ddots \\ b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & \ddots \\ b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & \ddots \\ b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & \ddots \\ b_{4,0} & b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & \ddots \\ b_{5,0} & b_{5,1} & b_{5,2} & b_{5,3} & b_{5,4} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{matrix}$$

Frequency subspaces W_ξ (non-truncated)

$$W_\xi := W_\xi^{(\alpha)} := \operatorname{clos}(\operatorname{span}\{m_{p,q}: \ p-q=\xi\}),$$

$$W_\xi = \operatorname{clos}(\operatorname{span}\{b_{p,q}: \ p-q=\xi\}).$$

$$\begin{array}{cccccc} m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & m_{0,4} & \ddots \\ m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & \ddots \\ \boxed{m_{2,0}} & m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & \ddots \\ m_{3,0} & \boxed{m_{3,1}} & m_{3,2} & m_{3,3} & m_{3,4} & \ddots \\ m_{4,0} & m_{4,1} & \boxed{m_{4,2}} & m_{4,3} & m_{4,4} & \ddots \\ \ddots & \ddots & \ddots & \boxed{\ddots} & \ddots & \ddots \end{array} \qquad \begin{array}{cccccc} b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & b_{0,4} & \ddots \\ b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & \ddots \\ \boxed{b_{2,0}} & b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & \ddots \\ b_{3,0} & \boxed{b_{3,1}} & b_{3,2} & b_{3,3} & b_{3,4} & \ddots \\ b_{4,0} & b_{4,1} & \boxed{b_{4,2}} & b_{4,3} & b_{4,4} & \ddots \\ \ddots & \ddots & \ddots & \boxed{\ddots} & \ddots & \ddots \end{array}$$

Description of the subspaces W_ξ and $W_{\xi,s}$ in polar coordinates

Here we use polar coordinates: $r \in [0, 1)$, $t \in \mathbb{T}$.

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W_ξ consists of all functions of the form

$$f(r\,t) = t^\xi\, r^{|\xi|}\, h(r^2), \qquad h \in L^2([0,1),\, (\alpha+1)(1-x)^\alpha\, x^{|\xi|}\,\mathrm{d}x).$$

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$W_{\xi,s}$ consists of all functions of the form

$$f(r\,t) = t^\xi\, r^{|\xi|}\, h(r^2), \qquad \text{where } h \text{ is a polynomial, } \deg(h) < s.$$

Fourier decomposition of $L^2(\mathbb{D}, \mu_\alpha)$

$$L^2(\mathbb{D}, \mu_\alpha) = \bigoplus_{\xi \in \mathbb{Z}} W_\xi^{(\alpha)}.$$

Fourier decomposition of $\mathcal{A}_n^2$

Proposition

$$\mathcal{A}_n^2 = \bigoplus_{\xi=-n+1}^{\infty} W_{\xi,\min\{n,n+\xi\}}.$$

$b_{0,0}$	$b_{0,1}$	$b_{0,2}$	$b_{0,3}$	$\cdots$
$b_{1,0}$	$b_{1,1}$	$b_{1,2}$	$b_{1,3}$	$\cdots$
$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$\cdots$
$b_{3,0}$	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Outline

To study radial operators in $\mathcal{A}_n^2$,
we use the following kindergarten result about von Neumann algebras.

It would be interesting to find generalizations of this result
or its analogs in the representation theory.

Definition: set of operators diagonalized by a family of subspaces

Let H be a Hilbert space, $\mathcal{U}$ be a self-adjoint subset of $\mathcal{B}(H)$,
and $(W_j)_{j\in J}$ be a countable family of nonzero closed subspaces of H such that

$$H = \bigoplus_{j\in J} W_j.$$

We say that this family diagonalizes $\mathcal{U}$ if the following two conditions are satisfied.

1. $\forall j \in J \qquad \forall U \in \mathcal{U} \qquad \exists \lambda_{U,j} \in \mathbb{C} \qquad W_j \subseteq \ker(\lambda_{U,j} I - U)$, i.e.,

$$\forall v \in W_j \qquad U(v) = \lambda_{U,j} v.$$

2. $\forall j, k \in J \qquad (j \neq k) \qquad \Longrightarrow \qquad \exists U \in \mathcal{U} \qquad \lambda_{U,j} \neq \lambda_{U,k}.$

Proposition

Let H, $\mathcal{U}$, and $(W_j)_{j\in J}$ be as in the definition.

Then the commutant of $\mathcal{U}$ consists of all bounded linear operators that act invariantly on each of the subspaces W_j, with $j \in J$:

$$\mathcal{U}' = \{S \in \mathcal{B}(H): \quad \forall j \in J \quad S(W_j) \subseteq W_j\}.$$

Furthermore,

$$\mathcal{U}' \cong \bigoplus_{j\in J} \mathcal{B}(W_j), \qquad W^*\text{-alg}(\mathcal{U}) = \mathcal{U}'' \cong \bigoplus_{j\in J} \mathbb{C} I_{W_j}.$$

Proposition

Let H, $\mathcal{U}$, and $(W_j)_{j\in J}$ be as in the definition,
and H_1 be a closed subspace of H invariant under $\mathcal{U}$.

For every U in $\mathcal{U}$, denote by $U|_{H_1}^{H_1}$ the compression of U onto H_1, and put

$$\mathcal{U}_1 := \left\{U|_{H_1}^{H_1}\colon\ U\in\mathcal{U}\right\},\qquad J_1 := \left\{j\in J\colon\ W_j\cap H_1\neq\{0\}\right\}.$$

Then

$$H_1 = \bigoplus_{j\in J_1}(W_j\cap H_1),$$

and the family $(W_j\cap H_1)_{j\in J}$ diagonalizes $\mathcal{U}_1$.

Outline

Rotation operators and radial operators

For each τ in $\mathbb{T}$, we denote by $\rho(\tau)$ the operator acting in $L^2(\mathbb{D}, \mu_\alpha)$ by

$$(\rho(\tau)f)(z) := f(\tau^{-1}z).$$

The family $\rho = (\rho(\tau))_{\tau\in\mathbb{T}}$ is a unitary representation of $\mathbb{T}$ in $L^2(\mathbb{D}, \mu_\alpha)$.

Radial operators in $L^2(\mathbb{D}, \mu_\alpha)$:

$$\mathcal{R} := \rho'.$$

Lemma

The family $(W_\xi)_{\xi\in\mathbb{Z}}$ diagonalizes the collection $\{\rho(\tau)\colon\ \tau\in\mathbb{T}\}$.

Lemma

The family $(W_\xi)_{\xi\in\mathbb{Z}}$ diagonalizes the collection $\{\rho(\tau):\ \tau\in\mathbb{T}\}$.

The proof is similar to a usual reasoning for rotations in $\mathcal{A}^2$. See Zorboska (2003).

Lemma

The family $(W_\xi)_{\xi\in\mathbb{Z}}$ diagonalizes the collection $\{\rho(\tau)\colon\ \tau\in\mathbb{T}\}$.

The proof is similar to a usual reasoning for rotations in $\mathcal{A}^2$. See Zorboska (2003).

1. Let τ in $\mathbb{T}$ and $\xi\in\mathbb{Z}$.

$$p-q=\xi \qquad \Longrightarrow \qquad \rho(\tau)b_{p,q}=\tau^{q-p}b_{p,q}=\tau^{-\xi}b_{p,q}.$$

$$W_\xi\subseteq\ker\bigl(\tau^{-\xi}I-\rho(\tau)\bigr).$$

Lemma

The family $(W_\xi)_{\xi\in\mathbb{Z}}$ diagonalizes the collection $\{\rho(\tau)\colon\ \tau\in\mathbb{T}\}$.

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$$p-q=\xi \qquad\Longrightarrow\qquad \rho(\tau)b_{p,q}=\tau^{q-p}b_{p,q}=\tau^{-\xi}b_{p,q}.$$

$$W_\xi\subseteq\ker\bigl(\tau^{-\xi}I-\rho(\tau)\bigr).$$

2. Let $\xi_1,\xi_2\in\mathbb{Z}$ and $\xi_1\neq\xi_2$.

$$\tau:=\exp\frac{\mathrm{i}\,\pi}{\xi_1-\xi_2},\qquad \tau^{-\xi_1}\neq\tau^{-\xi_2}.$$

Description of radial operators in $L^2(\mathbb{D}, \mu_\alpha)$

Theorem

$\mathcal{R}$ consists of all bounded linear operators that act invariantly on W_ξ for each ξ:

$$\mathcal{R} \;=\; \Big\{ S \in \mathcal{B}(L^2(\mathbb{D}, \mu_\alpha)) \colon \qquad \forall \xi \in \mathbb{Z} \qquad S(W_\xi) \subseteq W_\xi \Big\}.$$

Thereby,

$$\mathcal{R} \cong \bigoplus_{\xi \in \mathbb{Z}} \mathcal{B}(W_\xi).$$

Radial operators on $\mathcal{A}_n^2$

Rotations compressed to $\mathcal{A}_n^2$:

$$(\rho_n(\tau)f)(z) := f(\tau^{-1}z) \qquad (f \in \mathcal{A}_n^2).$$

Radial operators in $\mathcal{A}_n^2$:

$$\mathcal{R}_n := \rho_n'.$$

To apply the simple scheme, we compute the following intersections of subspaces:

$$W_\xi \cap \mathcal{A}_n^2 = \begin{cases} W_{\xi,\min\{n,n+\xi\}}, & \xi \geq -n+1, \\ \{0\}, & \xi < -n+1. \end{cases}$$

Description of radial operators in $\mathcal{A}_n^2$

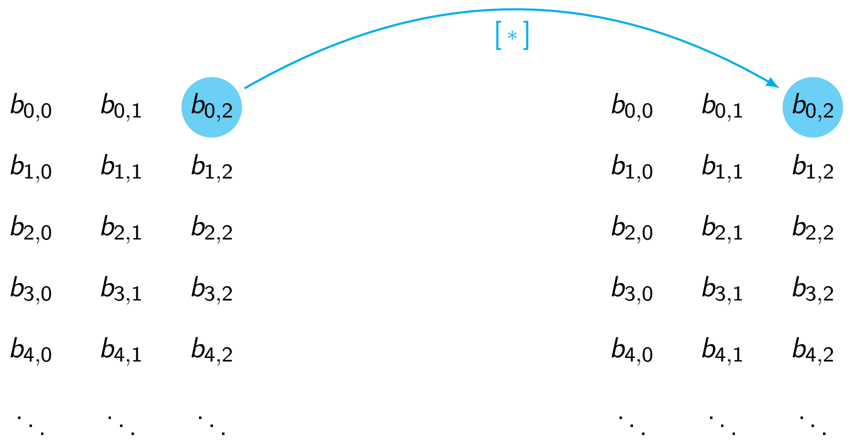
Theorem

$\mathcal{R}_n$ consists of all bounded linear operators in $\mathcal{A}_n^2$ that act invariantly on $W_{\xi,\min\{n,n+\xi\}}$ for each ξ.

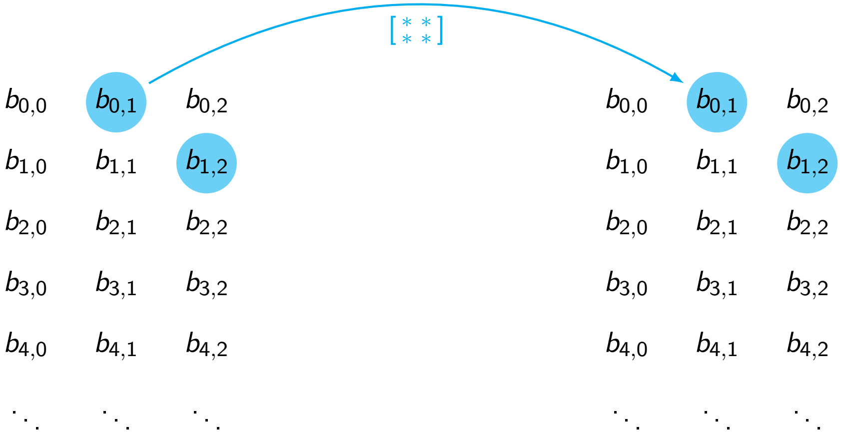
Thereby,

$$\mathcal{R} \cong \bigoplus_{\xi=-n+1}^{\infty} \mathcal{B}(W_{\xi,\min\{n,n+\xi\}}) \cong \bigoplus_{\xi=-n+1}^{\infty} \mathcal{M}_{\min\{n,n+\xi\}}.$$

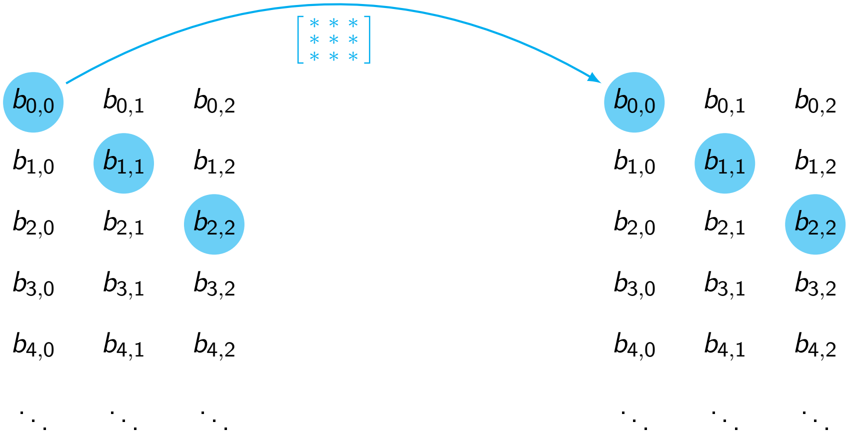
Radial operators act invariantly on the truncated frequency subspaces



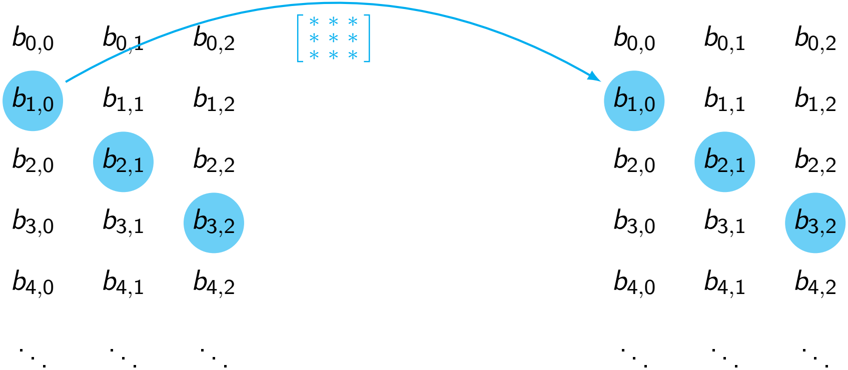
Radial operators act invariantly on the truncated frequency subspaces



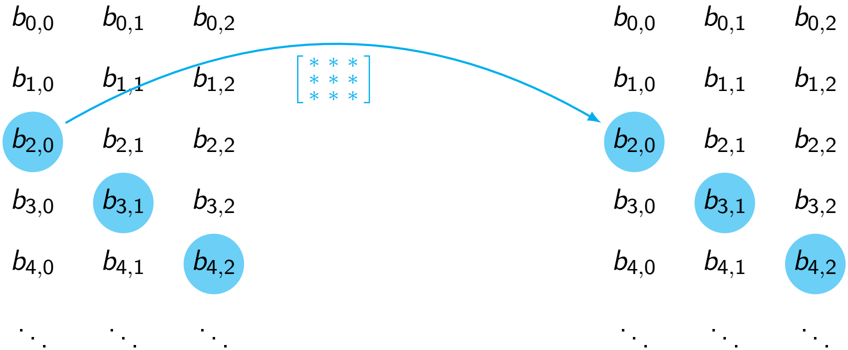
Radial operators act invariantly on the truncated frequency subspaces



Radial operators act invariantly on the truncated frequency subspaces



Radial operators act invariantly on the truncated frequency subspaces



Descomposition of $\mathcal{R}_n$

$$\mathcal{R}_n \cong \bigoplus_{\xi=-n+1}^{\infty} \mathcal{M}_{\min\{n,n+\xi\}}$$

$$\mathcal{R}_3 \cong \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \dots$$

$$\begin{bmatrix} [\,*\,] & & & & & \\ & \begin{bmatrix} * & * \\ * & * \end{bmatrix} & & & & \\ & & \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} & & & \\ & & & \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} & & \\ & & & & \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} & \\ & & & & & \vdots \end{bmatrix}$$

Outline

Radial Toeplitz operators correspond to radial generating functions

$T_{n,g}$:= Toeplitz operator in $\mathcal{A}_n^2$ with generating symbol g.

$$T_{n,g} \text{ is radial} \qquad \Longleftrightarrow \qquad g \text{ is radial.}$$

For $a \in L^\infty([0,1))$, we define $\widetilde{a} \in L^\infty(\mathbb{D})$ by

$$\widetilde{a}(z) := a(|z|).$$

The corresponding Toeplitz operator $T_{n,\widetilde{a}}$ is radial.

Sequence of matrices corresponding to a radial Toeplitz operator

Denote by $\gamma_n(a)$ the sequence of matrices corresponding to $T_{n,\widetilde{a}}$:

$$\underbrace{\begin{bmatrix} * \end{bmatrix}}_{\gamma_3(a)_{-2}}, \quad \underbrace{\begin{bmatrix} * & * \\ * & * \end{bmatrix}}_{\gamma_3(a)_{-1}}, \quad \underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{\gamma_3(a)_0}, \quad \underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{\gamma_3(a)_1}, \quad \underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{\gamma_3(a)_2}, \quad \underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{\gamma_3(a)_3}, \quad \ldots$$

$$\gamma_n(a)_\xi := \left[\beta_{a,\alpha,\xi,j,k}\right]_{j,k=\max\{0,-\xi\}}^{n-1},$$

$$\begin{aligned}\beta_{a,\alpha,\xi,j,k} &:= \langle T_{n,\widetilde{a}} b^{(\alpha)}_{k+\xi,k}, b^{(\alpha)}_{j+\xi,j} \rangle \\ &= \operatorname{coef}_{\alpha,|\xi|,j,k} \int_0^1 a(\sqrt{x})\, Q^{(\alpha,|\xi|)}_{\min\{j,j+\xi\}}(x)\, Q^{(\alpha,|\xi|)}_{\min\{k,k+\xi\}}(x)\, (1-x)^\alpha x^{|\xi|}\, \mathrm{d}x.\end{aligned}$$

Radial Toeplitz operators with limits at the boundary

Let RBL be the set of all bounded radial functions having limits at the boundary:

$$\mathrm{RBL} := \Big\{ \widetilde{a} \colon \quad a \in L^\infty([0,1)), \quad \exists L \in \mathbb{C} \quad \lim_{r \to 1} a(r) = L \Big\}.$$

Conjecture

The C*-algebra generated by

$$\{ \gamma_n(a) \colon \ a \in \mathrm{RBL} \}$$

is the C*-algebra of all matrix sequences having scalar limits.

Similar results for the vertical and angular case (when the generating symbols have limits at the boundary)

- Loaiza, Lozano (2014): On Toeplitz operators on the weighted harmonic Bergman space on the upper half-plane.
- Sánchez-Nungaray, González-Flores, López-Martínez, Arroyo-Neri (2018): Toeplitz operators with horizontal symbols acting on the poly-Fock spaces.
- Ramírez Ortega, Ramírez Mora, Sánchez Nungaray (2019): Toeplitz operators with vert. symbols acting on the poly-Bergman spaces of the upper half-plane. II.

Idea of the proof in these papers: separate pure states,
then apply Kaplansky's theorem (a non-commutative Stone-Weierstrass theorem).

Radial Toeplitz operators without limits at the boundary

Optimistic conjecture

If $a \in L^\infty([0,1))$, then there exists a log-slowly-oscillating sequence λ such that

$$\lim_{\xi\to\infty} \|\gamma_n(a)_\xi - \lambda_\xi I_n\| = 0.$$

$$\gamma_n(a)_\xi \quad \overset{\xi\to\infty}{\approx}$$