Radial Toeplitz operators on the Fock space and square-root-slowly oscillating sequences

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Outline

Outline

Bargmann–Segal–Fock space

$\mathrm{d}\mu_G(z) := \frac{1}{\pi}\, \mathrm{e}^{-|z|^2}\, \mathrm{d}\mu(z)$ is the Gaussian measure on the complex plane.

$$\mathcal{F}^2(\mathbb{C}) := L^2_a(\mathbb{C}, \mathrm{d}\mu_G), \qquad \text{i.e.,} \qquad \mathcal{F}^2(\mathbb{C}) := \{f \in H(\mathbb{C})\colon \quad \|f\|_{2,\mu_G} < +\infty\}.$$

$b_n(z) := \dfrac{z^n}{\sqrt{n!}}$, $\qquad (b_n)_{n\in\mathbb{N}_0}$ is an orthonormal basis of the Hilbert space $\mathcal{F}^2(\mathbb{C})$.

Mean-value theorem for analytic functions $\implies$ $\mathcal{F}^2(\mathbb{C})$ is a RKHS.

Reproducing kernel: $\qquad \mathcal{K}_z(w) = \sum_{n=0}^{\infty} \overline{b_n(z)}\, b_n(w) = \mathrm{e}^{\overline{z}\, w}.$

Rotation operators

$\mathbb{T} := \{z \in \mathbb{C}\colon\ |z| = 1\}$ is the unit circle in the complex plane.

For every τ in $\mathbb{T}$ and every f in $\mathcal{F}^2(\mathbb{C})$,

$$(\rho(\tau)f)(z) := f(\tau^{-1}z), \qquad \text{i.e.,} \qquad (\rho(\tau)f)(z) = f(\overline{\tau}\, z).$$

The family $\rho = (\rho(\tau))_{\tau\in\mathbb{T}}$ is a unitary representation of $\mathbb{T}$ in the Hilbert space $\mathcal{F}^2(\mathbb{C})$.

Three ways to verify that ρ is well defined, i.e., $\rho(\tau)\mathcal{F}^2 \subseteq \mathcal{F}^2$

1. $H(\mathbb{C})$ and μ_G are invariant under rotations.

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2. Action on the basis:

$$(\rho(\tau)b_n)(z) = b_n(\tau^{-1}z) = \frac{\tau^{-n}z^n}{\sqrt{n!}} = \tau^{-n}b_n(z),$$

$$\rho(\tau)b_n = \tau^{-n}b_n \in \mathcal{F}^2(\mathbb{C}).$$

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3. The RK is invariant under the simultaneous group action on both arguments:

$$\mathcal{K}_{\tau z}(\tau w) = \mathrm{e}^{\overline{\tau z}\,\tau w} = \mathcal{K}_z(w).$$

Radial operators on $\mathcal{F}^2(\mathbb{C})$

The von Neumann algebra of radial operators := the commutant of ρ.

$$\rho' := \{\rho(\tau)\colon\ \tau \in \mathbb{T}\}',$$

i.e.,

$$\rho = \Big\{S \in \mathcal{B}(\mathcal{F}^2(\mathbb{C}))\colon \quad \forall \tau \in \mathbb{T} \quad \rho(\tau)S = S\rho(\tau)\Big\}.$$

Radial operators = diagonal operators in the canonical basis

Recall that

$$\rho(\tau) b_n = \tau^{-n} b_n.$$

The representation ρ "separates the frequences":

$$\forall n, m \in \mathbb{N}_0 \qquad (m \neq n) \quad \Longrightarrow \quad (\exists \tau \in \mathbb{T} \quad \tau^{-m} \neq \tau^{-n}).$$

These properties imply that $\rho' =$ diagonal operators in the canonical basis.

$$\rho' \;\cong\; \bigoplus_{n=0}^{\infty} \mathbb{C} \;=\; \ell^{\infty}(\mathbb{N}_0).$$

Zorboska (2003).

Radial operators $\leftrightarrow$ bounded sequences

$$\lambda_n := \langle S b_n, b_n \rangle$$

$$S \in \rho' \quad \longrightarrow \quad \lambda \in \ell^\infty(\mathbb{N}_0)$$

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$$S b_n := \lambda_n b_n$$

Toeplitz operators

$P :=$ the orthogonal projection in $L^2(\mathbb{C}, \mu_G)$, whose image is $\mathcal{F}^2(\mathbb{C})$.

Given g in $L^\infty(\mathbb{C})$,

$$T_g f := P(g\, f).$$

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Given g in $L^\infty(\mathbb{C})$,

$$T_g f := P(g\, f).$$

Berger and Coburn (1987): the correspondence $g \mapsto T_g$ is injective.

Radial bounded functions

Given g in $L^\infty(\mathbb{C})$,

$$(\mathrm{rad}(g))(z) := \int_{\mathbb{T}} g(\tau z)\,\mathrm{d}\mu_{\mathbb{T}}(\tau) = \frac{1}{2\pi}\int_0^{2\pi} g(\mathrm{e}^{\mathrm{i}\vartheta} z)\,\mathrm{d}\vartheta.$$

Given a in $L^\infty([0,+\infty))$,

$$\widetilde{a}(z) := a(|z|).$$

Given g in $L^\infty(\mathbb{C})$, the following conditions are equivalent:

- $\forall \tau \in \mathbb{T} \quad \rho(\tau)g = g$;
- $g \overset{\mu-\text{a.e.}}{=\!=\!=} \mathrm{rad}(g)$;
- $\exists a \in L^\infty([0,+\infty)) \quad g = \widetilde{a}$.

Radial Toeplitz operators and their diagonalization

Given g in $L^\infty(\mathbb{C})$,

$$T_g \text{ is radial} \qquad \Longleftrightarrow \qquad g \text{ is radial.}$$

If $a \in L^\infty(\mathbb{R}_+)$ and $g = \widetilde{a}$, then

$$T_g b_n = \gamma_a(n) b_n,$$

where

$$\gamma_a(n) := \frac{1}{n!} \int_{\mathbb{R}_+} a(\sqrt{r}\,)\, \mathrm{e}^{-r}\, r^n \, \mathrm{d}r \qquad (n \in \mathbb{N}_0).$$

Grudsky and Vasilevski (2002).

Object of study: C*-algebra of radial Toeplitz operators

$$C^*(\{T_{\widetilde{a}}\colon\ a\in L^\infty(\mathbb{R}_+)\})\cong C^*(\mathcal{G}),$$

where

$$\mathcal{G}:=\{\gamma_a\colon\ a\in L^\infty(\mathbb{R}_+)\}.$$

Outline

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Main result: the sequences of the eigenvalues of radial Toeplitz operators
in the Bergman space on the unit ball in $\mathbb{C}^n$
generate the C*-algebra of the bounded log-oscillating sequences.

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Generalization to radial operators in the Bergman space on the weighted unit ball.

Bauer, W.; Herrera Yañez, C.; Vasilevski, N. (2014):
Eigenvalue characterization of radial operators on weighted Bergman spaces over the unit ball.
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More elementary proof (using convolutions):

Herrera Yañez, C.; Maximenko, E.A.; Vasilevski, N.L. (2015):
Radial Toeplitz operators revisited: Discretization of the vertical case.
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Recent results on the Toeplitz algebra

(without assuming symmetries)

Xia, J. (2015):
Localization and the Toeplitz algebra on the Bergman space.
https://doi.org/10.1016/j.jfa.2015.04.011

Bauer, W.; Fulsche, R. (2020):
Berger-Coburn theorem, localized operators, and the Toeplitz algebra.
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See also papers by Isralowitz, Mitkovski, Wick, and Hagger.

Outline

Square-root metric on $\mathbb{N}_0$

$\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

Define $\rho\colon \mathbb{N}_0 \times \mathbb{N}_0 \to [0, +\infty)$,

$$\rho(m, n) := \left|\sqrt{m} - \sqrt{n}\right|.$$

The modulus of continuity of a sequence σ with respect to ρ:

$$\omega_{\rho,\sigma}(\delta) := \sup\left\{|\sigma_n - \sigma_m|:\ \ m, n \in \mathbb{N}_0,\ \rho(m, n) \le \delta\right\}.$$

Square-root oscillating sequences

$\mathrm{RO}(\mathbb{N}_0)$:= bounded sequences, uniformly continuos with respect to ρ:

$$\mathrm{RO}(\mathbb{N}_0) := \left\{\sigma \in \ell^\infty(\mathbb{N}_0)\colon\ \lim_{\delta\to 0}\omega_{\rho,\sigma}(\delta) = 0\right\}.$$

In other words, $\sigma \in \mathrm{RO}(\mathbb{N}_0)$ iff $\sigma \in \ell^\infty(\mathbb{N}_0)$ and

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall m, n \in \mathbb{N}_0 \quad \Big(|\sqrt{m} - \sqrt{n}| < \delta \quad \Longrightarrow \quad |\sigma_m - \sigma_n| < \varepsilon\Big).$$

It is easy to see that $\mathrm{RO}(\mathbb{N}_0)$ is a C^*-subalgebra of $\ell^\infty(\mathbb{N}_0)$.

Examples and counter-examples of sqrt-oscillating sequences

Examples.

- $\sigma_n = \mathrm{e}^{\mathrm{i}\sqrt{n}}$.
- $\sigma_n = \cos(\sqrt{n})$.
- Converging sequences: $c(\mathbb{N}_0) \subsetneq \mathrm{RO}(\mathbb{N}_0)$.

Counter-examples.

- $\sigma_n = (-1)^n$. This oscillation is too fast.
- More generally, $\sigma_n = \tau^n$, with $\tau \in \mathbb{T} \setminus \{1\}$.

Lipschitz continuous sequences with respect to ρ

Proposition

Let $\sigma \in \ell^\infty(\mathbb{N}_0)$. Then the following conditions are equivalent:

- $\exists M > 0 \qquad \forall m, n \in \mathbb{N}_0 \qquad |\sigma_m - \sigma_n| \leq M|\sqrt{m} - \sqrt{n}|$;
- $\sup\limits_{n\in\mathbb{N}_0} \left(\sqrt{n+1}\, |\sigma(n+1) - \sigma(n)|\right) < +\infty.$

Idea of the proof:

$$\frac{1}{2}\sum_{j=n}^{m-1} \frac{1}{\sqrt{j+1}} \sim \sqrt{m} - \sqrt{n} \qquad (m, n \to \infty).$$

Extension of a sqrt-oscillating sequence to a sqrt-oscillating function

$\mathbb{R}_+ = [0, +\infty)$.

Idea: piecewise-linear interpolation with respect to the metric ρ.

Proposition

Let $\sigma \in \mathrm{RO}(\mathbb{N}_0)$. Define $f\colon \mathbb{R}_+ \to \mathbb{C}$ by the linear interpolation:

$$f(x) := \sigma_n + \frac{\sqrt{x} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}} (\sigma_{n+1} - \sigma_n), \qquad \text{where} \qquad n := \lfloor x \rfloor.$$

Then

$$f|_{\mathbb{N}_0} = \sigma, \qquad \|f\|_\infty = \|\sigma\|_\infty, \qquad f \in \mathrm{RO}(\mathbb{R}_+).$$

γ as an integral operator

Recall the formula for the eigenvalues of radial Toeplitz operators:

$$\gamma_a(n) = \frac{1}{n!}\int_{\mathbb{R}_+} a(\sqrt{r}\,)\,\mathrm{e}^{-r}\,r^n\,\mathrm{d}r = \frac{2}{n!}\int_{\mathbb{R}_+} a(y)\,\mathrm{e}^{-y^2}\,y^{2n+1}\,\mathrm{d}y \qquad (n \in \mathbb{N}_0).$$

It can be written as

$$\gamma_a(n) = \int_{\mathbb{R}_+} K(n,y)\,a(y)\,\mathrm{d}y,$$

where

$$K(n,y) := \frac{2y^{2n+1}}{n!\;\mathrm{e}^{y^2}}.$$

Metric on $\mathbb{N}_0$ with is “natural” for the eigenvalues’ sequences

Define $\varkappa\colon \mathbb{N}_0 \times \mathbb{N}_0 \to [0, +\infty)$,

$$\varkappa(m,n) := \sup_{\substack{a \in L^\infty(\mathbb{R}_+) \\ \|a\|_\infty = 1}} |\gamma_a(m) - \gamma_a(n)| \, .$$

Proposition

$$\varkappa(m,n) = \int_{\mathbb{R}_+} |K(m,y) - K(n,y)| \, \mathrm{d}y.$$

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Proposition

$$\varkappa(m,n) = \int_{\mathbb{R}_+} |K(m,y) - K(n,y)|\,\mathrm{d}y.$$

Idea of the proof: L^1 is the dual of L^∞.

In other words, we use a particular case of the Schur test for integral operators.

"Local behavior" of the natural metric

Proposition

$$\varkappa(n-1,n) = \frac{2n^n}{n!\ e^n}.$$

By Stirling formula,

$$\varkappa(n-1,n) \le \sqrt{\frac{2}{\pi n}}, \qquad \text{and} \qquad \varkappa(n-1,n) \sim \sqrt{\frac{2}{\pi n}} \quad \text{as} \quad n \to \infty.$$

Consequence (without proof): the intrinsic metric induced by $\varkappa$ is equivalent to ρ.

Eigenvalues' sequences are square-root oscillating

Theorem 1

$$\mathcal{G} \subseteq \mathrm{RO}(\mathbb{N}_0).$$

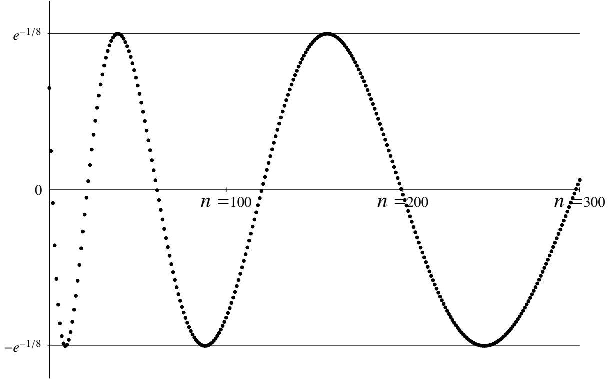
Proof. The inequality $\|\gamma_a\|_\infty \le \|a\|_\infty$ is trivial.

$$|\gamma_a(n) - \gamma_a(n-1)| \;\le\; \|a\|_\infty \varkappa(n, n-1) \;\le\; \sqrt{\frac{2}{\pi\, n}}\,\|a\|_\infty.$$

Thus, γ_a is Lipschitz continuous with respect to ρ.

Typical example: $a(y) := \cos(y)$

$$\gamma_a(n) = M(1+n, 1/2, -1/4) = \mathrm{e}^{-1/8}\cos(\sqrt{n}) + o(1), \qquad \text{as } n \to \infty.$$



Counter-example: rotation operator

Let $\tau \in \mathbb{T} \setminus \{1\}$.

Then $\rho(\tau) \in \rho'$.

The corresponding eigenvalues' sequence is

$$\sigma_n = \tau^{-n}.$$

Since $\sigma \notin \mathrm{RO}(\mathbb{N}_0)$, the operator $\rho(\tau)$ does not belong to the C^*-algebra generated by radial Toeplitz operators.

Outline

Dirac sequences

A sequence $(h_n)_{n\in\mathbb{N}}$ in $L^1(\mathbb{R})$ is called a **Dirac sequence**, iff

- $\forall n \in \mathbb{N} \qquad \forall x \in \mathbb{R} \qquad h_n(x) \geq 0,$
- $\forall n \in \mathbb{N} \qquad \int_{\mathbb{R}} h_n(t)\,\mathrm{d}t = 1,$
- $\forall \delta > 0 \qquad \lim_{n\to+\infty} \int_{|x|>\delta} h_n(t)\,\mathrm{d}t = 0.$

Dirac sequence serve to approximate uniformly bounded functions

The following result is well known.

Proposition

Let $f \in C_{b,u}(\mathbb{R})$ and $(h_n)_{n\in\mathbb{N}}$ be a Dirac sequence. Then

$$\lim_{n\to\infty} \left\| f * h_n - f \right\|_\infty = 0.$$

A version of Wiener's density theorem

Proposition

Let $k \in L^1(\mathbb{R})$ satisfy Wiener's condition:

$$\forall t \in \mathbb{R} \qquad \widehat{k}(t) \neq 0.$$

Then $\{k * f\colon\ f \in L^\infty(\mathbb{R})\}$ is a dense subset of $C_{b,u}(\mathbb{R})$.

Proof: use a Dirac sequence such that $\widehat{h}_n$ have compact supports, and apply Wiener's Division Lemma.

Outline

$$\gamma_a(n) = \int_{\mathbb{R}_+} a(y)\, K(n,y)\,\mathrm{d}y, \qquad \text{where} \qquad K(n,y) := \frac{2y^{2n+1}}{n!\ \mathrm{e}^{y^2}}.$$

$$\gamma_a(n) = \int_{\mathbb{R}_+} a(y)\, K(n,y)\,\mathrm{d}y, \qquad \text{where} \qquad K(n,y) := \frac{2y^{2n+1}}{n!\ \mathrm{e}^{y^2}}.$$

Main lemma

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} \left| K(n,y) - H(\sqrt{n}-y) \right| \mathrm{d}y = 0, \qquad \text{where} \qquad H(w) := \left(\frac{2}{\pi}\right)^{1/2} \mathrm{e}^{-2w^2}.$$

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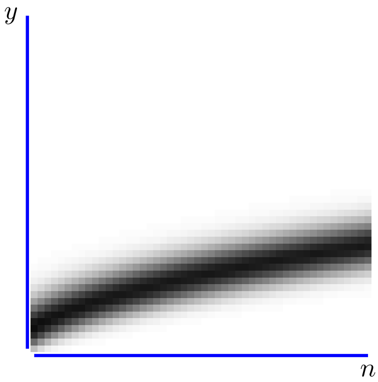
The heat kernel plays a crucial role in the theory of Toeplitz operators on Fock spaces.

$$H_t(w) := \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{|w|^2}{4t}}, \qquad H(w) = H_{1/8}(w).$$

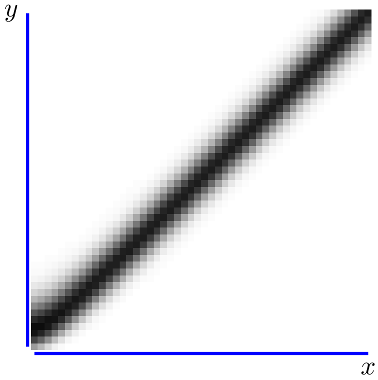
$$K(n,y) \approx H(\sqrt{n} - y).$$

The picture shows some values of $K(n,y)$. White points correspond to small values.



$$K(n,y) \approx H(\sqrt{n} - y).$$

The picture shows some values of $K(x^2, y)$. White points correspond to small values.



The tails of the eigenvalues' sequences can be approximated by convolutions

Proposition

Let $a \in L^\infty(\mathbb{R}_+)$. Then

$$\lim_{n\to+\infty} \left| \gamma_a(n) - \int_{\mathbb{R}_+} H(\sqrt{n} - y) a(y) \, \mathrm{d}y \right| = 0.$$

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Follows immediately from the main lemma.

The tails of the square-root-oscillating sequences can be approximated by eigenvalues' sequences

Proposition

Let $\sigma \in \mathrm{RO}(\mathbb{N}_0)$ and $\varepsilon > 0$. Then there exist $a \in L_\infty(\mathbb{R}_+)$ and $N \in \mathbb{N}$ such that

$$\sup_{n>N} |\sigma(n) - \gamma_a(n)| \leq \varepsilon.$$

The tails of the square-root-oscillating sequences can be approximated by eigenvalues' sequences

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$$\sup_{n>N} |\sigma(n) - \gamma_a(n)| \leq \varepsilon.$$

Proof: extend σ to a function $f \in \mathrm{RO}(\mathbb{R}_+)$,

then $h(x) := f(x^2)$ is uniformly continuous,

by Wiener's density theorem it can be approximated by a convolution with H.

Finally, we apply the previous proposition.

Eigenvalues of radial Toeplitz operators whose generating symbols have a limit at infinity

$$\mathcal{X} := \{u \in L^\infty(\mathbb{R}_+) : \quad \lim_{r\to+\infty} u(r) = 0\}.$$

Lemma

Let $u \in \mathcal{X}$. Then $\gamma_u \in c_0(\mathbb{N}_0)$.

This lemma is well known and easy to prove by direct estimates.

Approximation of the heads: dark evil witchcraft

Lemma

The set $\{\gamma_u\colon\ u \in \mathcal{X}\}$ is dense in $c_0(\mathbb{N}_0)$.

Our proof is not constructive. It is based on the Hahn–Banach theorem for $c_0(\mathbb{N}_0)$. We prove that the zero functional is the unique bounded linear functional on $c_0(\mathbb{N}_0)$ that annihilates the set $\{\gamma_u\colon\ u \in \mathcal{X}\}$.

In other words, we suppose that $p \in \ell^1(\mathbb{N}_0)$ such that

$$\forall u \in \mathcal{X} \qquad \sum_{n=0}^{\infty} \gamma_u(n) p_n = 0,$$

and we prove that $p = 0$.

Main result

Theorem 2

$\mathcal{G} = \{\gamma_a \colon\ a \in L^\infty(\mathbb{R}_+)\}$ is a dense subset of $\mathrm{RO}(\mathbb{N}_0)$.

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Proof:

- approximate by convolutions for large values of n,
- use the evil witchcraft for small values of n.

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Theorem 2

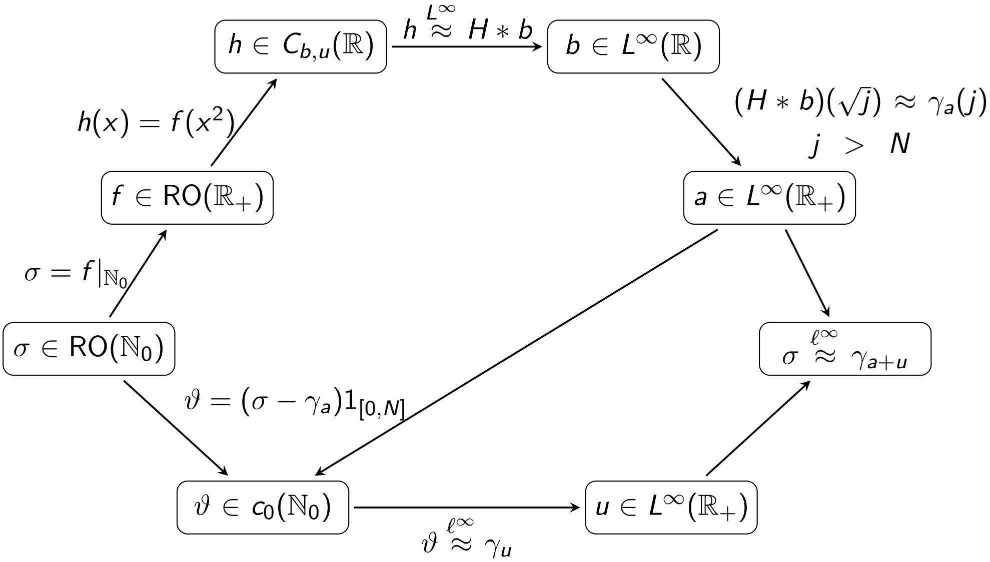
$\mathcal{G} = \{\gamma_a \colon\ a \in L^\infty(\mathbb{R}_+)\}$ is a dense subset of $\mathrm{RO}(\mathbb{N}_0)$.

Proof:

- approximate by convolutions for large values of n,
- use the evil witchcraft for small values of n.

Corollary (thanks to professor Ólafsson)

$\mathcal{G}$ is a weak-* dense subset of $\mathrm{RO}(\mathbb{N}_0)$.


$h \in C_{b,u}(\mathbb{R})$
$h \overset{L^\infty}{\approx} H * b$
$b \in L^\infty(\mathbb{R})$
$(H * b)(\sqrt{j}) \approx \gamma_a(j)$
$j > N$
$h(x) = f(x^2)$
$f \in \mathrm{RO}(\mathbb{R}_+)$
$a \in L^\infty(\mathbb{R}_+)$
$\sigma = f|_{\mathbb{N}_0}$
$\sigma \in \mathrm{RO}(\mathbb{N}_0)$
$\sigma \overset{\ell^\infty}{\approx} \gamma_{a+u}$
$\vartheta = (\sigma - \gamma_a)1_{[0,N]}$
$\vartheta \in c_0(\mathbb{N}_0)$
$u \in L^\infty(\mathbb{R}_+)$
$\vartheta \overset{\ell^\infty}{\approx} \gamma_u$